On the mathematical structure of Tonal Harmony

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Contents

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This work is dedicated to Euler [1], [2], [3] and Lagrange [4], [5], the fathers of Mathematical Physics of Music

II. NOTATION

$p_{Nature}(t)$	natural pound at time t
$p_{Culture}(t)$	cultural pound at time t
$\mathcal{R}_{ac}[i(t)]$	acoustical response to the input i(t)
$harmonic[\nu, n]$	n^{th} harmonic of the note ν
$\mathbf{mi}(\nu_1\,,\,\nu_2)$	musical interval among ν_1 and ν_2
ν_{ref}	reference note
$\mathbf{p}(\nu)$	pitch of the note ν
$S(\nu)$	Fourier transform of the signal $s(t)$
$\mathcal{D}$	tranfer function of the human ear
$scale - range(\nu)$	scale-range of the note ν
R_{ν}	rescaling function to the scale-range of ν
$a\ =\ b\, modc^{\mathbb{Z}}$	a is congruent to b modulo powers of c
$[a]_{c^{\mathbb{Z}}}$	residue class of a modulo powers of c
$\mathbb{Z}_{c^{\mathbb{Z}}}$	set of the residue classes of a modulo powers of c
c_n	n level Euler coordination
$\mathcal{N}_{Euler}$	Euler's notes
$\mathcal{P}_{Euler}(\nu)$	Euler's point of ν
$\mathcal{S}_{Euler}$	Euler's space
H_{prime}	prime vector of extended Euler's space
$\mathcal{N}_{Euler}^{just-tuned}$	just-tuned Euler's notes
$\mathcal{S}_{Euler}^{just-tuned}$	just-tuned subset of Euler's space
$\mathcal{N}_{Euler}^{Pyt-tuned}$	pytagorically-tuned Euler's notes
$\mathcal{S}_{Euler}^{Pyt-tuned}$	pytagorically-tuned subset of Euler's space
$\mathcal{N}_{Euler}^{n-temp-tuned}$	n-tempered-tuned Euler's notes
$\mathcal{S}_{Euler}^{n-temp-tuned}$	n-tempered-tuned subset of Euler's space
$\mathcal{N}_{Euler}^{n_1,n_2,n_3-temp-tuned}$	n_1, n_2, n_3 -tempered-tuned Euler's notes
$\mathcal{S}_{Euler}^{n_1,n_2,n_3-temp-tuned}$	n_1, n_2, n_3- tempered-tuned subset of Euler's space
$\mathbb{E}_{Euler}^{notes}$	canonical notes' basis of Euler's space
$\mathbb{E}_{Euler}^{int}$	canonical intervals' basis of Euler's space

$\hat{K}f$	fifth comma (pytagorical comma)
$\hat{K}t$	third comma (syntonic comma)
Kf	pitch of the fifth comma
Kt	pitch of the third comma
$G_{Euler}(\nu_1\,,\,\nu_2)$	Euler's gradus suavitatis of $\nu_1\,,\,\nu_2$
$I[s_1\,\cdots\,s_n]$	index of physical consonance of $s_1\,,\cdots\,,\,s_n$
$\omega_1 \sim_{\mathbb{Q}}\,\cdots\,\sim_{\mathbb{Q}} \omega_n$	commensurability of $\omega_1\,,\cdots\,,\,\omega_n$
$I[\omega_1\,\cdots\,\omega_n\,\|\bar{a}]$	index of physical consonance among $\omega_1\,,\cdots\,,\,\omega_n$ w.r.t. $\bar{a}$
$scale(\omega\,,\,R)$	scale of ω at fixed interval R
$ord(\{a_n\})$	ordered permutation of $\{a_n\}$
$\downarrow t$	downward closure of the tree t
$TC(s)$	transitive closure of the set s
support(s)	support of the set s
V_{afa}	class of the pure sets
G	nodes' set of the graph **G**
$\rightarrow_G$	edges' binary relation of the graph **G**
$GRAPHS$	proper class all graphs
$D(\mathbf{G})$	decoration of the graph **G**
Σ^n	words of length n on the alphabet Σ
$\Sigma^\star$	words on the alphabet Σ
Σ^∞	sequences on the alphabet Σ
Σ^n_{NR}	nonrepetitive words of length n on the alphabet Σ
$\Sigma^\star_{NR}$	nonrepetitive words on the alphabet Σ
$\vec{x}$	word
λ	empty word
$\|\vec{x}\|$	length of the word $\vec{x}$
$string(n)$	n^{th} word in quasilexicographic order
$\|n\|$	length of the n^{th} word in quasilexicographic order
$\bar{x}$	sequence
$<_p$	prefix order relation
$\cdot$	concatenation operator

x_n	n^{th} digit of the word $\vec{x}$ or of the sequence $\bar{x}$
$\vec{x}(n)$	prefix of length n of the word $\vec{x}$ or of the sequence $\bar{x}$
$\vec{x}(n,m)$	subword of the sequence $\bar{x}$ obtained taking the digits from the n^{th} to the m^{th}
$\vec{x}^n$	word made of n repetitions of the word $\vec{x}$
$[i]_{12}$	i^{th} letter of the musical alphabet $\mathbb{Z}_{12}$
$C\,,\cdots,\,B$	musical notation for, respectively, $[0]_{12}\,,\cdots,\,[11]_{12}$
T_y	translation operator by y
Inv	inversion operator
$mode(\vec{x}\,,\,i)$	mode of the word $\vec{x}$ of i^{th} degree
$chord(\vec{x}\,,\,i\,,\,n)$	chord of $\vec{x}$ of i^{th} degree at level n
$maxlevel(\vec{x})$	maximum possible level of $\vec{x}$'s chords
$\hat{g}$	map on words induced by the map g on letters
$\sim_T$	translational equivalence relation on words
$\sim_{Inv}$	inversion's equivalence relation on words
$d(x,y)$	distance among x and y
$I(\vec{x})$	interval vector of $\vec{x}$
$\mathcal{S}_{greg}$	gregorian words
$\mathcal{S}_{cl}$	classical words
$\mathcal{T}_{Maz}$	Mazzola tonalities
$tonality(\vec{x}\,,\,n)$	tonality of the word $\vec{x}$ at level n
$\mathcal{T}_n$	tonalities at level n
$\mathcal{T}$	tonalities
$\mathcal{HW}(t)$	harmonic words of t
$\mathcal{P}(t_1,t_2)$	pivotal degrees of t_1,t_2
$\mathcal{C}(t\,,\,\mathcal{T}_{context})$	cadences of t w.r.t. to the context $\mathcal{T}_{context}$
$\mathcal{T}_{n.c.}(\vec{x},n)$	natural context for $tonality(\vec{x},n)$
$\mathcal{MC}(t\,,\,\mathcal{T}_{context})$	minimal cadences of t w.r.t. to the context $\mathcal{T}_{context}$

$SYM(\vec{x})$	symmetry group of the word $\vec{x}$
$\mathcal{M}_{Maz}(t_1\,,\,t_2)$	Mazzola's modulations from t_1 to t_2
$\mathcal{M}(t_1\,,\,t_2)$	modulations from t_1 to t_2
$\mathcal{MP}_{Maz}$	Mazzola's tonal musical pieces
$\mathcal{MP}$	tonal musical pieces
Ξ	pytagoric musical alphabet
$[i]_{12}^{(n)}$	i^{th} letter at the n^{th} cycle of Ξ
$C^{(n)}\,,\,\cdots\,,\,B^{(n)}$	musical notation for, respectively, $[0]_{12}^{(n)}\,,\,\cdots\,,\,[11]_{12}^{(n)}$
$[i]_{12}^{(n)\,just}$	i^{th} letter at the n^{th} cycle of of the just-intonation alphabet
$C_{just}^{(n)}\,,\,\cdots\,,\,B_{just}^{(n)}$	musical notation for, respectively, $[0]_{12}^{(n)\,just}\,,\,\cdots\,,\,[11]_{12}^{(n)\,just}$
T_z^{Pyt}	pytagoric translation by z
Inv^{Pyt}	pytagoric inversion
C_+	fifth cycle's raising
C_-	fifth cycle's lowering
$\sim_{T^{Pyt}}$	pytagoric translation's equivalence relation
$\tilde{x}$	pytagoric word
$\tilde{pw}(\vec{x},n)$	pytagoric word of $\vec{x}$ at cycle n
$\mathcal{T}^{Pyt}$	pytagoric tonalities
$\mathcal{T}_n$	pytagoric tonalities at level n
$\mathcal{HW}^{Pyt}(t)$	pytagoric harmonic words of t
$\mathcal{P}^{Pyt}(t_1,t_2)$	pytagoric pivotal degrees of t_1,t_2
$\mathcal{C}^{Pyt}(t\,,\,\mathcal{T}_{context}^{Pyt})$	pytagoric cadences of t w.r.t. to the context $\mathcal{T}_{context}^{Pyt}$
$\mathcal{MC}^{Pyt}(t\,,\,\mathcal{T}_{context}^{Pyt})$	pytagoric minimal cadences of t w.r.t. to the context $\mathcal{T}_{context}^{Pyt}$
$\mathcal{M}^{Pyt}(t_1\,,\,t_2)$	pytagoric modulations from t_1 to t_2
$\mathcal{M}_K^{Pyt}(t_1\,,\,t_2)$	pytagoric comma-modulations from t_1 to t_2
$\mathcal{MP}^{Pyt}$	pytagoric tonal musical pieces
$t_1 \sim_K t_2$	t_1 is a comma-displacement of t_2
$s_1 \sim_P s_2$	s_1 is P-metamere to s_2
$\sim_P s$	P-valence of s

III. INTRODUCTION

No example could be given of the radical dichotomy existing nowadays among Science and Humanities as the intellectual analysis on the structure of Contemporary Music.

Let us start analyzing the overwhelming confusion existing in most of the discussions concerning the concept of *musical consonance*.

The correct conceptual approach would consist in distinguishing among the following two notions:

- *physical consonance*: a known phenomenon in Acoustics concerning a particular mathematical structure of some patterns of sounds played simultaneously

- *esthetic consonance*: the set of esthetic rules codified by a given Harmomy, i.e. by a given human formalization of the admissible patterns of simultaneous sounds

As a matter of principle, no reason exist in assuming that *esthetic consonance* has to be constrained by *physical consonance*:

is the natural phenomenon of animals tearing to pieces themselves in the jungle an index of the fact that we should make the same ?

Indeed a great peculiarity of the human specie consists in that it has developed *Culture*: a certain (continously evolving in time) amount of information.

Our behavior is, conseguentially, determined nowadays both by Nature and by Culture, with relative proportions p_{Nature} and $p_{Culture}$ evolving with time as:

$$p_{Nature}(t) + p_{Culture}(t) = 1 \tag{3.1}$$

$$\frac{dp_{Nature}}{dt} < 0 \tag{3.2}$$

$$\frac{dp_{Culture}}{dt} > 0 \tag{3.3}$$

The estimation of the contributions of p_{Nature} and $p_{Culture}$ is, typically, a matter of great discussion among researchers, each one tending to over-estimate the role of its own research area.

It should be said, with this regard, that the colossal amount of sloveliness shown by "human scientists" whenever they try to adopt precise, rigorously defined scientific notions [6], [7] streghtens scientists' common opinion that nothing serious is going on in their exponential productions of words without an underlying rigorous mathematical language.

This state of affairs occurs, first of all, in all the analyses by Human Scientists concerning the old question: " is music [1] a language "?

Let us observe, with this regard, that a misconception of the mathematical notion of language, i.e. the semantical side of Mathematical-Logic based on the notion of model of a formal system about which I demand to the Appendix G.5 "Formal Logic" of [9] or to the 10^{th} chapter "Geometric Logic and Classifying Topoi" of [10], has led some musicologists (cfr. e.g. the second chapter "L'occidente cristiano e l'idea di musica" of the first part "I problemi estetici e storici della musica" of [11]) to see a mutual exlusivity among conceptions based on the link *music-poetry* (with an emphasis to the emotional valence of this latter) and conceptions based on the link *music-mathematics* (with an emphasis to formal aspects); since the semantical side of Mathematics allows, as a matter of principle, to encode both *intrinsic musical meanings* and *extrinsic musical meanings* [12]. such a misconception lacks of any conceptual ground.

An ever more catastrophic situation occurs, anyway, as to the application of *Information Theory* to *Esthetics*:

that the *esthetic level* of an art of work is someway related to its *amount of information* is something very intuitive; the lack of mathematical rigor in many attempt to apply Information Theory to Esthetics [13] [2] kept the scientific community away from the whole matter.

This is a pity since, whenever applied in a serious way [3], Information Theory could allow to get some insight, e.g. through the following footsteps:

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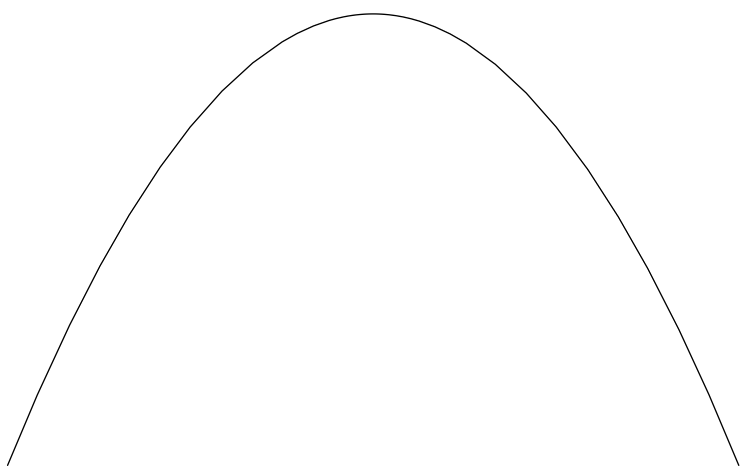
[1] The idle issue concerning the definition of the term music has led to curious theoretizations (we could call them *mafious esthetics*) such as the one expressed in [8] according to which the esthetical value of a work of art depends on whether the artist has paid the required percentage to the suitable corporation that conseguentially attributed to him the social status of artist,this fact determining, for example, that $4'33''$ by John Cage, whose score is : " I TACET ; II TACET ; III TACET" is an avant-garde masterpiece instead of being a fraud; according to these curious esthetics Bach, Mozart and Beethoven weren't great composer owing to the fact they produced great music: they composed great music, by definition, owing to the fact that they were composers

[2] and more generally the plethora of nonsense written by "social scientists" on speaking about information and codes (cfr. e.g. the cap.1 "L'universo dei segnali" and the cap.2 "L'universo del senso" of [14] or the cap.5 "La famiglia dei codici" of [15] based on the ravings about codes and Eco's s-codes of [16])

[3] A particular analysis should be made as to the section 3.10 "Peculiaritá e struttura statistica del messaggio musicale" of [17] where an inconsistent dichotomy among the *techno-semantic component* and the *esthetic component* of information is introduced

1. *Shannon's probabilistic information* has to be adopted to quantify the information's amount of an art of work relative to some cultural data, its absolute, intrinsic information's amount being quantified by its *Kolmogorov-Solomonoff-Chaitin's algorithmic information* [18]; for example the cultural data of Tonal Harmony may be codified as a probability distribution on the space of monophonic musical pieces w.r.t. which the pieces containing chromatic notes not belonging to the diatonic scale of the underlying tonality are strongly less probable than pieces not containing alterations and have, conseguentially an higher *probabilistic information* relative to Tonal Harmony though their absolute, *algorithmic information* is not higher [4].

2. the esthetical value of an art of work, seen as a function of its amount of information I, has the following qualitative behavior:

   i.e. it is very low as $I \approx 0$ (e.g. no one would consider the telephone "free-signal" as a musical masterpiece), it increases monotonically as I increases reaching a maximum after which it becomes to decrease tending to zero for maximal information (no one would consider a random art of work an esthetical masterpiece).

   Since also the *physical complexity* of an object, quantified by Bennett's notion of *logical depth* [20], [21] have such a qualitative behavior as a function of the information, one could be tempted to conjecture to consider *physical complexity* as an approximate, empirical quantitative measure of *esthetical value*, giving a firm foundation to the many speculations about the inter-relation among Esthetics and Complexity (as to

[4] A good example of such a kind of information-theoretical esthetical estimation is given by Gerard Hoffnung's not too flattering cartoon about Webern's music[19]

Music consider, for example, the debate about the complexity of contemporary music and Xenakis' notion of *mass* [22])

Returning to the issue of estimating the relative pounds p_{Nature} , $p_{Culture}$ in eq.3.1 let us observe , furthermore, that the increasing rule of the cultural pound, anyway, cannot delete the fact that the production, the propagation and the reception of musical sounds are physical phenomena occurring in Nature and described by the Laws of Acoustics.

The confusion among the concepts of *esthetic consonance* and of *physical consonance* has unfortunately led the overwhelming majority of the existing literature to make confusion between two completelly different issues:

1. the issue whether the phenomenon of musical consonance has a physical ground
2. the issue whether a positive answer to the previous issue would imply constraints on the Harmony's rules as to *esthetic consonance*

The confusion among these two, logically independent issues has:

- led some scientists to claim that the fact that Atonal Music is based on a notion of *esthetic consonance* differing from *physical consonance* more than Tonal Music implies that Atonal Music is esthetically inferior than Tonal Music [23]
- led some musicians to forget that the codification of new Harmonies with new *esthetic consonances* cannot delete the fact that *physical consonance* is a physical phenomenon occurring in Nature [5]

It is interesting, at this point, to observe [27] that many of the supporters of both the naturality and the esthetical superiority of Tonal Orthodoxy identified in Andreas Werck-

[5] a celebrated exposition of this argument was expressed,curiously, in "Human Sciences" by Claude Levi-Strauss, the father of all the discussions concerning the antinomy Nature versus Culture, in the "Overture" of [24] where he critizes the attitude by Pierre Boulez and many "serial thinkers" , of forgetting and deleting the existence of what he calls *"the first level of articulation"*; it should be observed, with this regard, that such a charge cannot be ascribed to Arnold Schönberg himself who clearly expressed, cfr. the cap.12 "Valutazione apollinea di un epoca dionisiaca" of [25] and the here cited analysis contained in the 7^{th} section "Sospensione ed eliminazione della tonalitá" and the 8^{th} section "La scala cromatica come fondamento della tonalitá" of the 19^{th} chapter "Alcune aggiunte e schemi che integrano il sistema" of [26], that his theory of dissonance's emancipation considered dissonances as farer consonances in the sequences of harmonics

meister's [28] introduction [6] of the chromatic equally-tempered scale the first semen of the atonal heresy, as it is condensed in the following sentence by Paul Hindemith about the discovery of the equable-tempering:

"Anyone who has ever tasted the delights of pure intonation by the continual displacement of the comma in string-quartet playing [7] , must come to the conclusion that there can be no such thing as atonal music, in which the existence of tone relationship is denied. The decline in the value placed upon tonality is based on the system of equal temperament, a compromise which is presented to us by the keyboard as an aid in mastering the tonal world, and then pretends to be the world itself. One needs only to have seen how the most fanatical lower of the piano will close his ears in horror at the falseness of the tempered chords of his instrument, once he has compared them a few times with those produced by a harmonium in pure intonation, to realize that with the blessing of equal temperament there entered into the world of music - lest the bliss of musical mortals be complete - a curse as well: the curse of too easy achievement of tone-connections. The tremendous growth of piano music in the last century is attributable to it, and in the "atonal style" I see its final fullfillment - the uncritical idolatry of tempered tuning"; (cited from chapter 4 "Harmony", section 10 "Atonality and Polytonality" of [31])

It is precisely this point that makes the whole subject interesting.

To show the inconsistence of argumentations, such as Molly Gustin's one [32] waved in [23], claiming to furnish a mathematical proof of the esthetical superiority of Tonal Music on Atonal Music would be a trivial and non interesting matter if it didn't lead to the further conceptual step of appreciating that:

1. Classical Tonal Harmony is itself invalidated by a mathematical inconsistence

[6] We adhere here, for simplicity, to the common attribution; from an historical perspective, anyway, we recall that the final Werckmeister's codification was significantly preceded by a plethora of other authors (cfr. e.g. the third section "Necessitá del temperamento" of the third chapter "I rapporti fra i suoni" of [29])

[7] As remarked by Pierce [30] both the thesis here supported by Hindemith according to which the players of arc-strings instruments adopt natural intervals in their solos as well as the thesis often opposed to it, i.e. the thesis claiming that they adopt well-tempered intervals seems to have been falsificated by concrete measurements: measurements show that the players of arc-strings instruments adopt nor natural neither well-tempered intervals

2. such a mathematical inconsistence affects the same formalization of the concept of Tonal Music,

3. the issue is deeply linked with the causes and the effects of Werckmeister's revolution

4. it corroborates Schönberg original viewpoint according to which Atonal Music was not a revolution but simply the last step in an historical process begun centuries before

IV. ON THE MUSICAL RELATIVITY THEORY

The passage from Special Relativity to General Relativity [33] may be formalized as the passage from the:

AXIOM IV.1

PRINCIPLE OF SPECIAL RELATIVITY

All the laws of Physics are the same in all the inertial frames

to the:

AXIOM IV.2

PRINCIPLE OF GENERAL RELATIVITY

All the laws of Physics are the same in all the frames

Each of the two principles leads, in the respective theory, to the corollaries:

Corollary IV.1

COROLLARY OF SPECIAL NON-INFERABILITY

No experiment allows an observer enclosed in a box to infer which is the particular inertial frame of the box

Corollary IV.2

COROLLARY OF GENERAL NON-INFERABILITY

No experiment allows an observer enclosed in a box to infer which is the particular frame of the box

Let us now suppose to pass from Physics to Music through the following:

TABLE OF ANSATZS: $PHYSICS \mapsto MUSIC$:

Physics	Harmony
frame	scale
inertial frame	diatonic scale

One obtains two musical theories consisting in a set of contraints about the laws of harmony, the Special Theory of Musical Relativity and the General Theory of Musical Relativity [8], based, respectively, on the following principles:

AXIOM IV.3

PRINCIPLE OF SPECIAL MUSICAL RELATIVITY

All the laws of Harmony are the same in all the diatonic scales

and:

AXIOM IV.4

PRINCIPLE OF GENERAL MUSICAL RELATIVITY

All the laws of Harmony are the same in all the scales

Driven by the analogy one could be immediately tempted to conjecture that axiomIV.3 and axiomIV.4 imply the following:

Corollary IV.3

COROLLARY OF SPECIAL MUSICAL NON-INFERABILITY

A listener cannot infer the particular diatonic scale of a tonal musical piece

Corollary IV.4

COROLLARY OF GENERAL MUSICAL NON-INFERABILITY

A listener cannot infer the scale of a musical piece

[8] these theories shouldn't be confused with the Henry Cowell's "Musical Relativity Theory" exposed in 1930 by the author in [34] (curiously the stimulating locution "Musical Relativity Theory" wasn't used by Cowell as a title for any of his many writings [35]) that constitutes a completely different thing about which I will return later

A conceptual bug, anyway, inficiates the passage from axiomIV.3 and axiomIV.4 to, respectively, corollaryIV.3 and corollaryIV.4: it implicitely assumes that the natural pound p_{Nature} of eq.3.1, as to the net effect on the listener, vanishes, a fact that we know to be false.

Let us leave aside, for a moment, such a conceptual bug precisely in order of looking where it would take us.

The adoption of the Special Relativity Theory and, hence, the imposition of corollaryIV.4 would then consist in the revolution realized by Andreas Werckmeister's equable-tempering who deleted the particular "colours" of the different tonalities [9] and was, for this reason, strongly opposed; as it is well known a determinant factor to the victory of Werckmeister's ideas was the support they received [10] from Johann Sebastian Bach's *Well-tempered Clavier* each of whose two books (respectively BWV 846-869 AND BWV 870-893 [11]) is made of 24 couple of preludes and fugues in each of the 24 ionic and aeolian tonalities .

Exactly as the passage from Special Relativity to General Relativity consists in deleting the existence of a special, privileged class of frames (the inertial ones), the passage from Special Musical Relativity to General Musical Relativity would be the departure from Tonal Harmony consisting in deleting the existence of a special, privileged class of scales.

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[9] E.g., as reported by H. Keller in the section "Il carattere della tonalitá nel Clavicembalo Ben Temperato" [36], D Major was thought to be the tonality in which arcs are most shining, G minor was associated in Baroque to the pompous music of official situations, the preferred tonality for pastoral music was F major in the North and G major in the Sud etc. (for a more detailed historical analysis cfr. the fourth section "Differenze espressive nel mondo tonale" of the fifth chapter "Tra senso e metafisica" of [29])

[10] It must be recalled, anyway, that from an historical point of view, it is not clear whether Bach referred to the equable-tempering, to the third, usually denoted as Werckmeister-III, of the six temperings introduced by Werckmeister or to something else (cfr. the section 7.3 "Scale temperate" of [23] and the third section "Necessitá del temperamento" of the third chapter "I rapporti fra i suoni" of [29])

[11] such an indexing of the tonalities has led Piergiorgio Odifreddi, in the title of his divulgative article [37], to talk of their "well-numbering"; such an indexing is, obviously, recursive, its concrete implementation on computer being given by the Miscellaneous - Music standard package of Mathematica [38] as well as by the Mathematica expressions PlayScale and PlayChord described in [39]. One could indeed think to go further comparing the notions of well-tempering and well-ordering; since the tempering operation consists in requiring the closure of the circle of fifths after 12 steps, i.e. in altering the dynamics on the circle transforming it from a quasi-periodic one to a periodic one by distributing, in a suitable way, the pytagorical comma (the difference among 12 pytagorical fifths and 7 octaves, equal to the difference among enharmonic notes in the pytagorical scale), such a procedure results in the passage from a non-numerable infinity of different tonalities to a finite number of different tonalities; while, before tempering, a well-ordering of the different tonalities is possible only assuming the Axiom of Choice, i.e. in an intrinsically non-constructive way [40], the tempering's procedure transforms the well-ordering of different tonalities into a constructive, and as we have seen even recursive [41], business

A first step in this direction follows Guerino Mazzola's observation [42] [9] [43] that the mathematical structure underlying the concept of tonality, whose intuitive content may be summarized as:

1. the existence, at each instant of time, of a tonal centre around which the musical discourse gravitates

2. the changes of the tonal centres are performed by Modulation Theory codifying them as symmetry transformations

is well larger than the classical notion of tonality characterizing 24 particular ways of gravitating around the tonal centre; a first enlargement of the notion of tonality consists in passing from the notion of *classical-tonality* to that of *gregorian-tonality* considering all the 84 modes of gregorian music as distinct tonalities; the elimination of any constraint in the characterization of the way of gravitating around the tonal centre leads to a further radical enlargement of the notion of tonality obtained introducing the notion of *Mazzola's tonality* and taking into account all the 792 Mazzola's tonalities.

A partial passage from the Special Musical Relativity Theory to the General Musical Relativity Theory would then consist in replacing the axiomIV.3 with an analogous Principle of Special Musical Relativity referred to such a generalized definition of the notion of tonality.

It must be observed, with this regard, that nobody has composed yet a sort of "Generalize well-tempered clavier", i.e. a collection of two books of ordered musical composition, each book consisting of an ordered collection of musical compositions in all these possible "generalized tonalities".

As I will show later, anyway, the notion of Mazzola tonality may itself be generalized further.

Returning now to the mentioned conceptual bug, it allows immediately to explain why cor.IV.3 is trivially false:

our psycho-acoustic perception of sound is absolute and not relative: the *level of sonorous sensation (phon)*, the *level of subjective sonorous sensation (son)* , the *critical band*, the *subjective height (mel)* and many other phycho-acustic physical quantities depend on the frequency (cfr. e.g. the cap.4 of [23], the cap.7 of [30] and the cap. 3 of [44]): so it is not surprising that the phenomenon of *perfect pitch* (the ability of some people to identify a

single note without a reference to an external diapason) is an experimental evidence that falsificates cor.IV.3.

The contribution to p_{Nature} of all these psycho-acoustic factors is, anyway, strongly lower than the contribution of an external acoustic factor: the role of *harmonics*, whose discussion requires a brief report on the foundations of Acoustics.

V. ELEMENTS OF ACOUSTICS

Demanding to [44],[45], [30], [23] for any further information let us recall that from a physical point of view, a *sonore signal* is codified by the associated-function $s : \mathbb{R} \mapsto \mathbb{R}$ such that s(t) expresses the ratio among the perturbation at time t produced by the *sonore signal* in the pressure of a fixed point of the propagating medium and the Pascal (the unity measure of pression in the International System I will assume from here and beyond).

A *sonore signal* s(t) may be seen, more precisely, as the *acoustical response*:

$$s[t] \; = \; \mathcal{R}_{ac}[i(t)] \tag{5.1}$$

(in pression) of the medium to an input $i(t)$ produced by some *sonore source*.

The functional *acoustical response* $\mathcal{R}_{ac}$ is usually, nonlinear; beside in the phenomenon of *combinational notes* I will discuss in section VI the deviation from linearity of the *acoustical response* is neglibible, having no physical relevance [12].

It is then natural, in these cases, to assume the validity of the following:

AXIOM V.1

PRINCIPLE OF SUPERPOSITION:

$$\mathcal{R}_{ac}[\lambda_1 i_1(t) + \cdots + \lambda_n i_n(t)] \; = \; \lambda_1 \mathcal{R}_{ac}[i_1(t)] + \cdots \lambda_n \mathcal{R}_{ac}[i_n(t)] \tag{5.2}$$

A sonore signal will be said to be a *sound* if its *associated function* is periodic.

the *note* of a *sound* of period T is then defined as the *frequency* of its associated function $\nu := \frac{1}{T}$.

Given two notes ν_1 and ν_2 with $\nu_1 \leq \nu_2$

DEFINITION V.1

MUSICAL INTERVAL AMONG ν_1 and ν_2:

$$\mathbf{mi}(\nu_1 \, , \, \nu_2) \; := \; \frac{\nu_2}{\nu_1} \tag{5.3}$$

[12] The introduced distinction among linear and nonlinear acoustic response is, obviously, of a more general nature, as it can be appreciated introducing the mechanical and electric equivalents of an acoustic system and looking at non-linear acoustic response functions in terms of resistors with non-linear characteristic (cfr. e.g. the section1.6 "Analogie elettriche, meccaniche ed acustiche" of [44] and the 10^{th} chapter "Differential equations for electrical circuits" of [46])

Fixed once and for all a reference note ν_{ref} (e.g. by the assumption $\nu_{ref} := C_2 := 132 Hz$ adopted in the appendix 2.3 "Frequency and Glissando " of [9]):

DEFINITION V.2

PITCH OF THE NOTE ν (IN CENTS):

$$\mathbf{p}(\nu) := \frac{1200}{\log(2)} \log(\mathbf{mi}(\nu , \nu_{ref})) \tag{5.4}$$

A *sound* of frequency ν is said to be *pure* if its associated function s(t) is a trigonometric function, i.e. it is of the form $s(t) = A \sin(2\pi\nu t)$ or $s(t) = A\cos(2\pi\nu t)$, the positive factor $A \in \mathbb{R}_+$ being called the *amplitude*.

By Fourier transform any *sonore signal* may be seen as the uncountable infinite superposition of *pure sounds* of any possible frequency:

Theorem V.1

FOURIER DECOMPOSITION OF A SONORE SIGNAL

$$s(t) = \int_0^\infty d\nu (a(\mu)\cos(2\pi\mu t) + b(2\pi\mu t)\sin(2\pi\mu t))\, d\mu \tag{5.5}$$

where the two functions:

$$a(\mu) = \int_{-\infty}^{+\infty} dt s(t)\cos(2\pi\mu t) \tag{5.6}$$

and

$$b(\mu) = \int_{-\infty}^{+\infty} dt s(t)\sin(2\pi\mu t) \tag{5.7}$$

are called the *Fourier components* of the *sonore signal*

If the *sonore signal* is itself a *sound* of *note* ν , furthermore, theoremV.1 immediately implies that:

1. only a countable infinity of *pure sounds* contribute to its *Fourier components*
2. the notes of this infinity of Fourier-contributing *pure-sounds* (called the *harmonics* of the sound) , are the multiple integers of the note of the considered sound

as is stated by the following:

Corollary V.1

HARMONIC DECOMPOSITION OF A SOUND:

$$a(\mu) \; = \; \sum_{n=0}^{\infty} a_n \delta(\nu - n\mu) \tag{5.8}$$

$$b(\mu) \; = \; \sum_{n=0}^{\infty} b_n \delta(\nu - n\mu) \tag{5.9}$$

It may be useful, a this point, to introduce the following:

DEFINITION V.3

HARMONIC SEQUENCE OF THE NOTE ν:

the sequence of notes $\{harmonic(\nu, n)\}_{\{ n \in \mathbb{N}_{+} \}}$:

$$harmonic(\nu, n) \; := \; (n+1)\,\nu \tag{5.10}$$

I will adopt, from here and beyond, the usual complex representation of harmonic motion, representing a sound $s(t) := \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n \omega t) + b(n) \sin(n \omega t))$ through the complex function:

$$\tilde{s}(t) \; := \; \sum_{n=-\infty}^{+\infty} c_n e^{i n \omega t} \tag{5.11}$$

where:

$$c_n \; := \; \begin{cases} \frac{a_0}{2}, & \text{if } n = 0; \\ \frac{a_n - i b_n}{2}, & \text{if } n > 0; \\ \frac{a_n + i b_n}{2}, & \text{if } n < 0 \end{cases} \tag{5.12}$$

dropping the tilde from here and beyond.

As to arbitrary sonore signals, furthermore, I will adopt, from here and bejond, the analogous complex representation of Fourier integrals:

$$s(t) \; = \; \int_{-\infty}^{+\infty} d\nu S(\nu) e^{2 \pi i \nu t} \tag{5.13}$$

$$S(\nu) \; = \; \int_{-\infty}^{+\infty} dt s(t) e^{-2 \pi i \nu t} \tag{5.14}$$

The concept that a *sound* is the composition of other sounds is the source of much confusion, that may be completelly avoided taking strongly into account the following remarks:

1. the *amplitude* of a *pure sound*, ruling the *intensity* of the involved pressure-wave, may be seen as a parameter measuring its *volume*

2. the only harmonic contributing to the harmonic decomposition of a *pure sound* with note ν is ν itself [13]

3. if the amplitudes of two *pure sounds* are, roughly speaking, of the same order of size, they are perceived by our ears as two distinct *sonore signals*.

4. if the amplitudes of two *pure sounds* are, roughly speaking, of different order of size, our ears perceive them as a unique *sonore signal* whose associate function is the sum of the associated functions of the two *pure sounds*

5. the reason why the *pure-sounds* contributing to the harmonic decomposition of a *non-pure sound* are perceived as a unique *sonore signal* is that the amplitude a_n and b_n decrease enough quickly as n grows, as is implied by the Bessel's inequality:

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \ \leq \ \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s^2(t)dt \ \ \forall n \in \mathbb{N} \tag{5.15}$$

so that, for different values of n, they are always of different order of size

The real situation is, anyway, more complex since:

1. any concretely occurring *sonore signal* has finite support,*sounds* are just a mathematical idealization not occurring in reality

2. one usually adopts the previously introduced metaphorical locution according to which our ears hear the harmonical components of a sonore signal; the human auditory anatomic behavior is not that of a computer getting from the ear device the input consisting in the whole specification of the function s(t) and after that computing its whole Fourier transform

These observations have led Guerino Mazzola (cfr. the 2^{th} chapter "Topography" and the 15^{th} part "Sound" of [9]), referring to Jean Molino's scheme concerning the three-parts a musical communication's stream is made of:

[13] Among all the physical nonsense of Schönberg's analysis about harmonics in the first section of the fourth chapter of [26] this is a point in which he makes trivially false statements ending up with the "harmonics of harmonics" and the involved regressum ad infinitum (SIC!)

Poiesis the production of music by a creator

Neutral Niveau the transmitted musical message

Esthesis the reception of the musical message by a listener

to contest the traditional viewpoint of Acoustics giving for granted that the notions of note and pitch belong to the neutral level.

Mazzola doesn't anyway seems to appreciate the role of the Theorem of Tonal Indetermination we are going to introduce.

Given a sonore signal s(t) let us introduce the following probability distributions over: $(\mathbb{R}\,,\,\mathcal{B}(\mathbb{R}))$ [14]:

$$P_s(t) \;:=\; \frac{|s(t)|^2}{E} \tag{5.16}$$

$$P_S(\nu) \;:=\; \frac{|S(\nu)|^2}{E} \tag{5.17}$$

where, owing to Parseval's equation, one has that:

$$E \;:=\; \int_{-\infty}^{+\infty} dt\,|s(t)|^2 \;=\; \int_{-\infty}^{+\infty} d\nu |S(\nu)|^2 \tag{5.18}$$

It may be proved that (cfr. the section 5.10.6 "Sul concetto di trasformazione" of the 5^{th} chapter "La trasmissione dell'informazione" of [17], the appendix N-3 "Analytic Signals and the Uncertainty Relation" of [47], the section 8.10 "The Uncertainty Principle" of [48], the section 19.4 "Wavelet transformation" of [49] and the part D "Wavelet Analysis" of [50]) that:

Theorem V.2

THEOREM OF NOTE'S INDETERMINATION:

HP:

$$s(t) \;=\; o_{t \rightarrow \pm \infty}(t) \tag{5.19}$$

TH:

[14] $\mathcal{B}(\mathbb{R})$ denoting the Borel-σ-algebra of $\mathbb{R}$

$$\Delta t \, \Delta \nu \;\geq\; \frac{1}{4\pi} \tag{5.20}$$

where Δt and $\Delta \nu$ are the standard-deviations of, respectively, the probability distributions 5.16 and 5.17.

The conceptual deepness of theoremV.2 was fully analyzed by Dennis Gabor who:

1. conjectured that is may be seen as a particular case $out(t) = in(t)$ of an information-theoretic indetermination's theorem generalizing to arbitrary channels with band-width ΔB such that:

$$supp(T(\omega)) \;=\; \Delta \nu \tag{5.21}$$

   Nyquist's criterium:

$$\Delta B \, \Delta t \;=\; constant \tag{5.22}$$

   on the minimum temporal distance Δt among two Dirac-delta's impulses such that they can be distinguished at the output of a Nyquist channel, i.e. a channel with transfer function:

$$T(\omega) \;:=\; \frac{Out(\omega)}{In(\omega)} \;=\; |A(\omega)| \, e^{-i\,\beta(\omega)} \tag{5.23}$$

   with:

$$A(\omega) \;=\; \begin{cases} A_0, & \text{if } |\omega| \leq \omega_0; \\ 0, & \text{otherwise.} \end{cases} \tag{5.24}$$

$$\sqrt{a}\beta(\omega) \;=\; \omega \, t_0 \tag{5.25}$$

2. understood its role in determining the invariance of the acquired information in any signal's analysis performed by a central computational unit (e.g. the human brain) with an inner time-clock Δt getting the sound s(t) through the sampling performed by a device (e.g. the human ear) with linear transfer function $\mathcal{D}_L$ [15] whose band-width is less or equal to the one admitted by Nyquist's criterium5.22 for sampling;

[15] Exactly as the acoustical response of the medium, the transfer function of the human ear may be decomposed in its linear and non linear parts:

$$\mathcal{D} \;=\; \mathcal{D}_L \,+\, \mathcal{D}_{NL} \tag{5.26}$$

$\mathcal{D}_{NL}$ conspires together with the nonlinear part of the medium acoustic response to generate the nonlinear effects such as *combinational tones* discussed in the section VI and many other effects about which

such an invariance follows, precisely, from the fact that the Gabor's transformation of the signal (that I am going to introduce) saturates eq.V.2, i.e. it respects it with the equal sign, so that its time-frequency representation is constitued of rectangular windows of linear dimensions $(2\sqrt{a}\,,\,\frac{1}{\sqrt{a}})$ and hence of constant area $A = 2$. The concept's of Gabor transformation is based on the idea that, if one is interested in getting the local-information about a signal localized around the time $t = b$ it is reasonable to generalize the Fourier's tranform introducing into it a localized gaussian weight-function centered around $t = b$. The generalization of this concept to a more general class of possible small localized weight-functions led Jean Morlet to introduce the following notions:

DEFINITION V.4

MOTHER WAVELET:

a function $\psi \in L^2(\mathbb{R})$ such that:

$$0 < \int_{-\infty}^{+\infty} \frac{|\Psi(\nu)|^2}{|\psi|} < +\infty \tag{5.28}$$

where, according to our general notation, $\Psi(\nu)$ denotes the Fourier transform of $\psi(t)$.

Given a mother wavelet ψ:

DEFINITION V.5

WAVELET GENERATED BY ψ:

---

we demand to the 3^{th} chapter "L'orecchio e la percezione del suono" of [44], to the appendix B "Auditory Physiolosy and Psychology" of [9] or the text [51] enterely ddvoted to such issue.Exactly as to the acoustcial response, we will ignore from here and beyond the physical effect involved with $\mathcal{D}_{NL}$ not being relevant for the issue discussed in this paper; we will furthermore refer to this approximation as the assumption of the following generalization of axiomV.1, i.e.:

AXIOM V.2

EXTENDED PRINCIPLE OF SUPERPOSITION:

$$\mathcal{D} \circ \mathcal{R}_{ac}[\lambda_1 i_1(t) + \cdots + \lambda_n i_n(t)] = \lambda_1 \mathcal{D} \circ \mathcal{R}_{ac}[i_1(t)] + \cdots \lambda_n \mathcal{D} \circ \mathcal{R}_{ac}[i_n(t)] \tag{5.27}$$

the two parameter family of functions $\{\psi_{a,b}(t)\}_{b \in \mathbb{R}, a \in \mathbb{R} - \{0\}}$ given by:

$$\psi_{a,b}(t) \ := \ \frac{1}{|a|^{\frac{1}{2}}} \, \psi(\frac{t-b}{a}) \tag{5.29}$$

DEFINITION V.6

WAVELET TRANSFORM W.R.T. ψ OF $f \in L^2(\mathbb{R})$:

$$T_{\psi}(a\,,\,b) \ := \ \int_{-\infty}^{+\infty} dt\, f(t) \psi^{\star}(t) \tag{5.30}$$

We can at last introduce the following:

DEFINITION V.7

GABOR TRANSFORMATION:

the *wavelet tranform* generated by a gaussian mother wavelet

It is important to stress that, though the Mathematics underlying theoremV.2 is exactly that of the particular application of the Heisenberg's Indetermination Theorem in Quantum Mechanics:

$$< \alpha | (\Delta \hat{A})^2 (\Delta \hat{B})^2 | \alpha > \geq \ \frac{1}{4} | < \alpha | [\hat{A}, \hat{B}] | \alpha > |^2 \tag{5.31}$$

to the couple of observables *position* and *momentum* $\hat{x}$, $\hat{p}$ [16], the issue is quite different from a physical point of view; it should be mentioned, with this regard, that the adoption of the quantities Δt and $\Delta \nu$ as measures of the uncertainty in, respectively, time and frequency of a sonore signal s(t) is someway arbitary, since no physical ground exists in the assumption of the involved probability, respectively, distributions $P_s(t)$ and $P_S(\nu)$ mathematically assumed by mimicking the probability distribution of quantum-mechanical wave-functions.

[16] I would like to advise the reader that in 5^{th} chapter "La trasmissione dell'informazione" of [17] it is erroneously stated that theoremV.2 implies the indetermination's relation *time* and *energy* therein referred as Heisenberg's Principle. As stressed by any elementary Quantum Mechanics' manual (cfr. e.g. the section 5.6 "Teoria perturbativa dipendente dal tempo" of [52]), anyway, the indetermination's relation *time* and *energy*, inferred in the framework of first-order perturbative theory, is conceptually radically different from Heisenberg's Principle; in particular, time is not an observable in Quantum Mechanics as it is significantly testified by the conceptual bugs one has to face on trying to define a *quantum clock* about which I demand to section 12.7 "The measurement of time" of [53], as well as to [54] and [55]

The standard deviation of $P_S(\nu)$ is, anyway, certainly a better measure of the uncertainty in frequency of s(t) than the semi-distance among its first two zeroes of a function only approximating $S(\nu)$ that has been adopted in the section 3.3 "Pacchetti d'onda. Indeterminazione tonale" of [23] on the basis of the section 12.7 "Pacchetti d'onda. Velocitá di gruppo " of [56].

Let us now return to the information-theoretical point of view to Music briefly introduced in sectionIII, taking into account, for simplicity, the production of a monofonic musical piece made of notes of the same duration:

such a musical piece may then be seen as a *string* or as a *sequence* on the alphabet $\mathbb{R}_+$ of all the possible notes.

Since:

$$card((\mathbb{R}_+)^\star) \ = \ 2^{card(\mathbb{R}_+)} \ = \ 2^{\aleph_1} \ = \ \aleph_2 \tag{5.32}$$

$$card((\mathbb{R}_+)^\infty) \ = \ 2^{card((\mathbb{R}_+)^\star)} \ = \ \aleph_3 \tag{5.33}$$

both considering terminating and nonterminating musical pieces one has that the set of all the possible musical messages has a cardinality higher than the continuum one, i.e. is made of "a more than continuously infinity" of possible musical executions with the consequential troubles as far as codification of musical messages is concerned.

This simple consideration allows to understand why, from the beginning of its structurally organized formalization of Music, humanity tried to limit the space of the possible musical messages introducing strongly smaller alphabets.

Different cultures pursued this goal in different ways: the structural similarity of the obtained results, anyway, is a very strong evidence of the role that the common underlying Laws of Nature had in such a process.

We have touched, at this point, the key argument of this paper enclosed in the following:

Question V.1

QUESTION ON THE "MORE NATURAL MUSICAL MESSAGES" DETERMINED BY THE HARMONIC SEQUENCES

At which extent does the "naturality" of the harmonic sequences determines the "naturality" of some particular musical messages ?

The naive answer to questionV.1 one usually reads many times is the following:

Answer V.1

THE NAIFE ANSWER:

The natural role played, in the Laws of Nature, by the first five harmonics determines the Nature-induced maximally consonant triad: the major triad

Since, as I will extensively discuss, the naife answerV.1 is false, let us proceed more carefully starting to discuss, first of all, the "natural" role of the first harmonic.

A strong empirical evidence of its pervasivity in different cultures strongly leads to assume the following:

AXIOM V.3

AXIOM OF PERCEPTION OF REPETITION FOR THE FIRST HARMONIC

the natural role played in the Laws of Nature by the first harmonic is such that, given a sequences of notes $\{\nu_n\}_{n \in \mathbb{N}}$, the sequence of notes $\{harmonic(\nu_n, 1)\}_{n \in \mathbb{N}}$ is perceived as a repetition of the sequence $\{\nu_n\}_{n \in \mathbb{N}}$ at an higher range

AxiomV.3 played a basic role in all the ways in which different cultures constructed *scale of notes* they adopted as musical alphabet; conseguentially, in all cultures, a basic role was played, in such a construction, by the following notion:

DEFINITION V.8

SCALE RANGE OF A NOTE ν:

$$scale - range(\nu) \;:=\; [\nu \, , \, harmonic(\nu, 1)] \tag{5.34}$$

Let us now observe, that assuming the axiomV.3, the construction of any kind of *scale of notes* starting from a note ν may be limited to its *scale range*: outside the *scale range* everything is simply repeated by multiplying each note of the considered scale for a suitable power of two.

Contrary, any note lying outside the scale range of ν may be rescaled to it by dividing it for a suitable power of two.

These operations may be easily performed through the following:

DEFINITION V.9

RESCALING FUNCTION TO THE SCALE-RANGE OF ν:

the function $R_{\nu} \; : \; \mathbb{R}_{+} - scale - range(\nu) \; \mapsto \; \mathbb{R}_{+}$ identified by the following constraints:

- for $\mu \; < \; \nu$:

$$R_{\nu}(\mu) \; = \; 2^{f_{\nu}(\mu)} \; \cdot \; \mu \tag{5.35}$$

with:

1. $$f_{\nu}(\mu) \; \in \; \mathbb{N} \tag{5.36}$$

2. $$2^{f_{\nu}(\mu)} \; \cdot \; \mu \; \geq \; \nu \tag{5.37}$$

3. $$2^{f_{\nu}(\mu)} \; \cdot \; \mu \; < \; 2\nu \tag{5.38}$$

- for $\mu \; > \; 2\nu$:

$$R_{\nu}(\mu) \; = \; 2^{- \, f_{\nu}(\mu)} \; \cdot \; \mu \tag{5.39}$$

1. $$f_{\nu}(\mu) \; \in \; \mathbb{N} \tag{5.40}$$

2. $$2^{- \, f_{\nu}(\mu)} \; \cdot \; \mu \; \geq \; \nu \tag{5.41}$$

3. $$2^{- \, f_{\nu}(\mu)} \; \cdot \; \mu \; < \; 2\nu \tag{5.42}$$

Some simple consideration leads to the following:

Lemma V.1

EXPLICIT FORMULA FOR THE RESCALING FUNCTION:

$$R_{\nu}(\mu) \; = \; \begin{cases} 2^{Int(\frac{\log \nu \, - \, \log \mu}{\log 2} \, + 1)} \; \cdot \; \mu & \text{if } \mu \; < \; \nu \; , \\ 2^{- \, Int(\frac{\log \mu \, - \, \log \nu}{\log 2})} \cdot \; \mu & \text{if } \mu \; > \; 2 \, \nu . \end{cases} \tag{5.43}$$

PROOF:

- for $\mu < \nu$:

  passing to the logarithms in the disequalities one obtains that:

$$f_{\nu}(\mu) \in [A_{\nu}(\mu) , A_{\nu}(\mu) + 1) \tag{5.44}$$

  where:

$$A_{\nu}(\mu) = \frac{\log \mu - \log \nu}{\log 2} \tag{5.45}$$

  The thesis immediately follows imposing the constraint that $f_{\nu}(\mu)$ has to be integer

- for $\mu > 2\nu$:

  passing to the logarithms in the disequalities one obtains that:

$$f_{\nu}(\mu) \in [A_{\nu}(\mu) , A_{\nu}(\mu) + 1) \tag{5.46}$$

  where:

$$A_{\nu}(\mu) = \frac{\log \mu - \log \nu}{\log 2} - 1 \tag{5.47}$$

  The thesis immediately follows imposing the constraint that $f_{\nu}(\mu)$ has to be integer

■

Both the *scale-range* of a note ν and the rescaling function to its scale-range may be computed through the following Mathematica's expressions:

```
harmonic[nu_,n_]:=(n+1)*nu


scalerange[nu_]:=Interval[nu,harmonic[nu,1]]


rescalingtorange[nu_,mu_]:= If[mu < nu,
    mu*Power[2, IntegerPart[Times[Log[mu] - Log[nu], Power[2, -1]]] +
1], If[mu>2nu,
mu*Power[2,-IntegerPart[Times[Log[mu]-Log[nu],Power[2,-1]]]],
"undefined"]]
```

taken from my Mathematica [38] noteboook *Mathematical Music Theory* reported in the sectionA

The rescaling function allows to introduce the following useful generalization of Congruence Theory [57]:

given three real numbers $a, b, c \in \mathbb{R}$:

DEFINITION V.10

a IS CONGRUENT TO b MODULO POWERS OF c ($a = b \, mod \, c^{\mathbb{Z}}$):

$$\exists n \in \mathbb{Z} \; : \; a = b \, mod \, c^{n} \tag{5.48}$$

DEFINITION V.11

RESIDUE CLASS OF a MODULO POWERS OF c:

$$[a]_{c^{\mathbb{Z}}} := \{ x \in \mathbb{R} \; : \; x = a \, mod \, c^{\mathbb{Z}} \} \tag{5.49}$$

For $c \in \mathbb{N}$ we can introduce, furthermore, the following:

DEFINITION V.12

SET OF THE RESIDUE CLASSES MODULO POWERS OF c:

$$\mathbb{Z}_{c^{\mathbb{Z}}} := [a]_{c^{\mathbb{Z}}} \; : \; a \in [0, c-1] \tag{5.50}$$

VI. HORIZONTAL AND VERTICAL RULES IN TONAL HARMONY

Tonal Harmony is made of two ingredients:

VERTICAL RULES IN SCORES The theory of physical consonance among sounds inside a fixed tonality, ruled according to the following:

AXIOM VI.1

AXIOM OF THE NATURALITY OF ESTHETICS:

$$\text{esthetic consonance} = \text{physical consonance} \tag{6.1}$$

The existence of a physical acoustical reason underlying the phenomonological evidence that certain couples of notes are perceived by our ears as consonant, i.e. the net acoustical input sounds good, while other couples of notes are perceived by our ears as dissonant,i.e. the net acoustical input sounds bad, was first suggested in 1638 by Galileo Galilei [58] in the following famous passage:

"SALVIATI: ⋯ Returning now to the original subject of discussion, I assert that the ratio of a musical interval is not immediately determined by the length, size, or tension of the strings, but rather by the ratio of their frequencies, that is, by the number of pulses of air waves which strike the tympanum of the ear, causing it also to vibrate vith the same frequency. This fact established, we may possibly explain why certain pairs of notes differing in pitch produce a pleasing sensation, other a less pleasant effect, and still others a disagreeable sensation. Such an explanation would be tantamount to an explanation of the more or less perfect consonances and of dissonances. The unpleasant sensation produced by the latter arises, I think, from the discordant vibrations of two different tones which strike the ear out of time. Especially harsch is the dissonance between notes whose frequencies are incommensurable; such a case occurs when one has two strings in unison and sounds one of them open, together with a part of the other which bears the same ratio to its whole length as a side of a square bears to the diagonal; this yelds a dissonance similar to the augmented fourth or diminished fifth.

Agreeable consonance are pairs of tones which strike the ear with a certain regularity; this regularity consists in the fact that the pulses delivered by the two notes, in the same interval of time, shall be commensurable in number, so as not keep the ear drum in perpetual torment, bending in two different directions in in order to yeld to the ever-discordant impulses ". from part 146-147, pagg.103-104 of [58]

We see that:

- Galileo didn't know the phenomenon of harmonics and made the error of thinking that strings produce pure sounds
- he identified the source of the physical consonance between these supposed pure sounds in the condition of commensurability of their frequencies (and the consequent regularity of the tympanum's sollecitation).

The first attempt of furnishing a quantitative measure of physical consonance was performed by L. Euler [1] through the introduction of a function *gradus suavitatis* expressing the degree of consonance among two notes in terms of the "simplicity" of their ratio. Even more remarkable, was ,anyway, the introduction of the space on which the *gradus suavitatis* is defined, that I will call, following Mazzola (cfr. the 15^{th} part "Sound" of [9]) the *Euler's space*.

Gixen an integer positive number $n \in \mathbb{N}$ and denoted by $\pi(i)$ the i^{th} prime number:

DEFINITION VI.1

n-LEVEL EULER'S COORDINATION

the map $c_n : \mathbb{R}^n \mapsto \mathbb{R}$:

$$c_n(x_1 \, , \, \cdots \, , \, x_n) \; := \; \prod_{i=1}^{n} \pi(i)^{x_i} \tag{6.2}$$

One has that:

Lemma VI.1

ON THE INJECTIVITY OF EULER'S COORDINATIONS:

1. c_n is not injective
2. $c_n|_{\mathbb{Q}^n}$ is injective

PROOF:

1. the set $c_n^{-1}(y)$ of the counterimages of a positive real $y \in [0\,,\,+\infty)$ is given by the solutions of the following equation:

$$\prod_{i=1}^{n} \pi(i)^{x_i} \;=\; y \tag{6.3}$$

i.e.:

$$\sum_{i=1}^{n} x_i \, \log \pi(i) \;=\; \log y \tag{6.4}$$

or:

$$x_2 \;=\; \frac{\log y \;-\; \sum_{i=1}^{n} x_i \, \log \pi(i)}{\log \pi(1)} \tag{6.5}$$

so that:

$$card(c_n^{-1}(y)) \;=\; \aleph_1 \;>\; 1 \tag{6.6}$$

2. let us proceed by induction:
   - proving the injectivity of C_2:
     since the integer solutions of the equations:

     $$2^{x_1} 3^{x_2} \;=\; 2^{y_1} 3^{y_2} \tag{6.7}$$

     are the same of the equation:

     $$\frac{x_1 \;-\; y_1}{x_2 \;-\; y_2} \;=\; -\, \log_2 3 \tag{6.8}$$

     the thesis immediately follows by the fact that $\frac{x_1 - y_1}{x_2 - y_2}$ is rational or undetermined while $-\log_2 3$ is irrational
   - proving that the injectivity of c_n is implied by the injectivity of c_{n-1}:
     since by the inductive hypothesis:

     $$card(c_{n-1}^{-1}(y)) \;\leq\; 1 \;\; \forall y \in [0, \infty) \tag{6.9}$$

     to prove that:

     $$card(c_n^{-1}(y)) \;\leq\; 1 \;\; \forall y \in [0, \infty) \tag{6.10}$$

it is sufficient to show that:

$$(\pi(n)^{x_n} \;=\; \pi(n)^{y_n}) \;\Leftrightarrow\; (x_n \;=\; y_n) \tag{6.11}$$

But this is a trivial consequence of the bijectivity of the exponential function

■

DEFINITION VI.2

EULER'S NOTES:

$$\mathcal{N}_{Euler} \;:=\; \{\nu_{ref}\, c_3(\vec{x})\,,\, \vec{x} \in \mathbb{Q}^3\} \tag{6.12}$$

Given an Euler's note ν the lemmaVI.1 allows to introduce the further notions:

DEFINITION VI.3

EULER'S POINT OF ν :

$$\mathcal{P}_{Euler}(\nu) \;:=\; c_3^{-1}(\frac{\nu}{\nu_{ref}}) \tag{6.13}$$

DEFINITION VI.4

EULER'S SPACE:

$$\mathcal{S}_{Euler} \;:=\; \mathcal{P}_{Euler}(\mathcal{N}_{Euler}) \;=\; \mathbb{Q}^3 \tag{6.14}$$

One has clearly that:

Theorem VI.1

1. $\mathcal{S}_{Euler}$ is a linear space on the field $\mathbb{Q}$

2.

$$\mathcal{P}_{Euler}(\nu_R) \;=\; (0\,,\,0\,,\,0) \tag{6.15}$$

$$\mathcal{P}_{Euler}(2\nu_R) \;=\; (1\,,\,0\,,\,0) \;=:\; \hat{o} \tag{6.16}$$

$$\mathcal{P}_{Euler}(\frac{3}{2}\nu_R) \ = \ (0\,,\,1\,,\,0) \ =: \ \hat{q} \tag{6.17}$$

$$\mathcal{P}_{Euler}(\frac{5}{4}\nu_R) \ = \ (0\,,\,0\,,\,1) \ = \ \hat{t} \tag{6.18}$$

The passage from points in Euler's space to the associated pitchs can, by construction, be obtained through the following:

Theorem VI.2

FROM EULER POINTS TO PITCHS:

$$\mathbf{p}(\nu) \ = \ \frac{1200}{\log(2)} H_{prime} \cdot (\mathcal{P}_{Euler}(\nu)) \tag{6.19}$$

where:

$$H_{prime} \ := \ (\log(2), \log(3), \log(5)) \tag{6.20}$$

is called the *prime vector*

Remark VI.1

ON THE PLANE ORTHOGONAL TO THE PRIME VECTOR IN THE EXTENDED EULER SPACE

Introduced the:

DEFINITION VI.5

EXTENDED EULER SPACE:

$$\mathcal{S}_{Euler}^{EXT} \ := \ \{x\hat{o} + y\hat{f} + z\hat{t} \ , x, y, z \in \mathbb{R}\} \ = \ \mathbb{R}^3 \tag{6.21}$$

the theoremVI.2 implies, obviously, that:

Corollary VI.1

$$\mathbf{p}(\nu) \;=\; H_{prime} \cdot (\mathcal{P}_{Euler}(\nu) + E) \;\; \forall E \in H_{prime}^{\perp} \, , \forall \nu \in \mathcal{N}_{Euler} \tag{6.22}$$

Since as a function from $\mathbb{R}_+$ to $\mathbb{R}$ $\mathbf{p}(\nu)$ is obviously an injective function, one could get a moment of surprise as to the corollaryVI.1 and erroneously be led to think that an Euler note is represented by many Euler points.

CorollaryVI.1 is, instead, simply nothing but a consequence of the lemmaVI.1 stating that Euler's coordination, defined on the extended Euler's space, is not injective

DEFINITION VI.6

CANONICAL NOTES' BASIS OF EULER SPACE:

$$\mathbb{E}_{Euler}^{notes} \;:=\; \{\hat{o} \, , \, \hat{f} \, , \, \hat{t}\} \tag{6.23}$$

The chosen names for versors in the definitionVI.6 is owed to the fact that:

1. $\hat{o}$ is the Euler point of the note being distant an interval of octave from ν_R
2. $\hat{f}$ is the Euler point of the note being distant an interval of perfect fifth from ν_R (both in pytagoric and in just tunings, concepts that we will introduce later)
3. $\hat{t}$ is the Euler point of the note being distant an interval of major third from ν_R (in just tuning, a concept we will introduce later)

Let us then consider some following important subsets of Euler notes and the corresponding subsets of representative Euler points in Euler's space:

DEFINITION VI.7

JUST-TUNED EULER NOTES:

$$\mathcal{N}_{Euler}^{just-tuned} \;:=\; \{\nu_{ref} \, c_3(\vec{x}) \, , \, \vec{x} \in \mathbb{Z}^3\} \tag{6.24}$$

DEFINITION VI.8

JUST-TUNED SUBSET OF EULER'S SPACE:

$$\mathcal{S}_{Euler}^{just-tuned} \;:=\; \mathcal{P}_{Euler}(\mathcal{N}_{Euler}^{just-tuned}) \;=\; \{x_o \hat{o} \,+\, x_f \hat{f} \,+\, x_t \hat{t} \,:\, x_o, x_f, x_t \in \mathbb{Z}\} \tag{6.25}$$

DEFINITION VI.9

PYTAGORICALLY-TUNED EULER NOTES:

$$\mathcal{N}_{Euler}^{Pyt-tuned} \;:=\; \{\nu_{ref}\, c_3(x_o \hat{o} \,+\, x_f \hat{f})\,,\, x_o, x_f \in \mathbb{Z}\} \tag{6.26}$$

DEFINITION VI.10

PYTAGORICALLY-TUNED SUBSET OF EULER'S SPACE:

$$\mathcal{S}_{Euler}^{Pyt-tuned} \;:=\; \mathcal{P}_{Euler}(\mathcal{N}_{Euler}^{Pyt-tuned}) \;=\; \{x_o \hat{o} \,+\, x_f \hat{f} \;:\; x_o, x_f \in \mathbb{Z}\} \tag{6.27}$$

Given an integer number $n \in \mathbb{N}$:

DEFINITION VI.11

n-TEMPERED TUNED EULER NOTES:

$$\mathcal{N}_{Euler}^{n-temp-tuned} \;:=\; \{\nu_{ref}\, c_3(\frac{x_o}{n} \hat{o})\,,\, x_o \in \mathbb{Z}\} \tag{6.28}$$

DEFINITION VI.12

n-TEMPERED-TUNED SUBSET OF EULER'S SPACE:

$$\mathcal{S}_{Euler}^{w-temp-tuned} \;:=\; \mathcal{P}_{Euler}(\mathcal{N}_{Euler}^{w-temp-tuned}) \;=\; \mathbb{Z}\frac{1}{n}\hat{o} \tag{6.29}$$

Given three integer numbers $n_1\,,\, n_2\,,\, n_3 \in \mathbb{N}$:

DEFINITION VI.13

n_1, n_2, n_3-TEMPERED-JUST-TUNED EULER NOTES

$$\mathcal{N}_{Euler}^{n_1,n_2,n_3-temp-just-tuned} \;:=\; \{\nu_{ref}\, c_3(\frac{x_o}{n_1}\hat{o} \,+\, \frac{x_f}{n_2}\hat{f} \,+\, \frac{x_t}{n_3}\hat{t})\,,\, x_o, x_f, x_t \in \mathbb{Z}\} \tag{6.30}$$

DEFINITION VI.14

n_1, n_2, n_3-TEMPERED-JUST-TUNED SUBSET OF EULER'S SPACE:

$$\mathcal{S}_{Euler}^{n_1,n_2,n_3-temp-just-tuned} := \mathcal{P}_{Euler}(\mathcal{N}_{Euler}^{n_1,n_2,n_3-temp-just-tuned}) \tag{6.31}$$

It is important, at this point, to observe that Euler's space $\mathcal{S}_{Euler}$ may be used not only to denote *notes* but *interval among notes*. With this respect is is then useful to introduce the following:

DEFINITION VI.15

CANONICAL INTERVALS' BASIS OF EULER SPACE:

$$\mathbb{E}_{Euler}^{int} := \{\hat{o}\,,\, \hat{f} - \hat{o}\,,\, \hat{t} - 2\hat{o}\} \tag{6.32}$$

whose elements represents, respectively, the *octave interval*, the *perfect fifth interval* and the *major third interval*.

Let us then introduce two intervals whose theoretical relevance will soon appear:

DEFINITION VI.16

FIFTH COMMA (PYTAGORICAL COMMA):

$$\hat{Kf} := -7\hat{o} + 12(\hat{f} - \hat{o}) \tag{6.33}$$

DEFINITION VI.17

THIRD COMMA (SYNTONIC COMMA):

$$\hat{Kt} := 2\hat{o} - 4(\hat{f} - \hat{o}) + (\hat{t} - 2\hat{o}) \tag{6.34}$$

Applying the theoremVI.2 one finds that:

$$\mathbf{Kf} := \mathbf{p}(\mathcal{P}_{Euler}^{-1}(\hat{Kf})) \approx 23.46\,Cents \tag{6.35}$$

$$\mathbf{Kf} := \mathbf{p}(\mathcal{P}_{Euler}^{-1}(\hat{Kf})) \approx -21.61\, Cents \tag{6.36}$$

Let us now analyze how all the previously discussed modulo-octave stuff appears in the framework of Euler space.

The computations about Euler's space may be performed using he following Mathematica expression of the mentioned notebook of sec.A:

```
eulercoordination[x_List]:=
  Product[Power[Prime[i],Part[x,i]], {i,1,Length[x]}]

FROMeulerpointTOnote[eulpoint_List] :=
  referencenote*2^Part[eulpoint,1]*3^Part[eulpoint,2]*5^Part[eulpoint,3]

FROMnoteTOpitch[nu_]:= Times[1200,Power[Log[2],-1]]
*Log[nu]-Log[Times[nu,Power[referencenote,-1]]]

FROMpitchTOnote[pitch_]:=
referencenote*Exp[Times[Log[2],Power[1200,-1]]*pitch]

Hprime=Table[Log[Prime[i]],{i,1,3}];

FROMeulerpointTOpitch[eulpoint_List]:=
Times[1200,Power[Log[2],-1]]*Dot[eulpoint,Hprime]

lastpartisequaltosomethingQ[x_,something_]:=Equal[Last[x],something]

rationalsupto[n_]:=Flatten[ Table[Rational[i,j],{i,0,n},{j,1,n}] ]

FROMindexTOrational[index_,n_]:=Part[rationalsupto[n],index]
```

```
FROMlistofindicesTOlistofrationals[list_,n_]:=
  Table[FROMindexTOrational[Part[list,i],n],{i,1,Length[list]}]

FROMnoteTOeulerpoint[nu_,n_]:=
  FROMlistofindicesTOlistofrationals[
    First[generalizedselect[
        Flatten[Table[ {{i,j,k},
              Dot[{Part[ rationalsupto[2],i ],Part[ rationalsupto[2],j],
                  Part[ rationalsupto[2],k ]},Hprime]},{i,1,
              Length[ rationalsupto[2] ]},{j,1,Length[ rationalsupto[2]
]},{k,
              1,Length[ rationalsupto[2] ]}],2]  ,
        lastpartisequaltosomethingQ,FROMnoteTOpitch[nu]]],n]

FROMwordTOlistofeulerpoints[w_]:=
  Table[FROMnoteTOeulerpoint[Part[FROMwordTOscale[w] ,i]],{i,1,Length[w]}]
octaveepoint={1,0,0}; fifthpoint={0,1,0}; thirdpoint={0,0,1};

canonicalnotesbasis:={octaveepoint,fifthpoint,thirdpoint}

canonicalintervalsbasis:={octaveepoint,fifthpoint-octaveepoint,
    thirdpoint-2octaveepoint}
```

Euler's formalization of the consonance's issue was based on the introduction of the following function:

given a positive integer number $x \in \mathbb{N}_+$, the Theorem of Prime Factorization assures the existence and the unicity of a sequence $\{e_n\}_{n \in \mathbb{N}_+}$ of positive integer numbers such that:

1. $\{e_n\}_{n \in \mathbb{N}_+}$ ends with a countable infinity of zeros, i.e.:

$$\exists N \in \mathbb{N}_+ : (e_n = 0 \, \forall n > N) \tag{6.37}$$

2.
$$x = \prod_{n=1}^{\infty} \pi(n)^{e(n)} \tag{6.38}$$

DEFINITION VI.18

GRADUS SUAVITATIS OF THE POSITIVE INTEGER x:

$$G_{Eul}(x) := 1 + \sum_{k=1}^{\infty} e_k(\pi(k) - 1) \tag{6.39}$$

Given a positive rational number $r \in \mathbb{Q}_+$ we have by definition that:

$$\exists!(p,q) \in \mathbb{N}_+^2 : (x = \frac{p}{q} \, and \, gcd(p,q) = 1) \tag{6.40}$$

We can consequentially generalize the definition VI.18 in the following way:

DEFINITION VI.19

GRADUS SUAVITATIS OF THE POSITIVE RATIONAL x:

$$G_{Eul}(x) := G_{Eul}(p \cdot q) \tag{6.41}$$

The definition VI.19 may be then applied to the ratio of Just-tuned Euler notes resulting in the following function

DEFINITION VI.20

GRADUS SUAVITATIS OF BICHORDS OF JUST TUNED EULER NOTES:

the map $G_{Eul} : \mathbb{N}_{Euler}^{just-tuned} \times \mathbb{N}_{Euler}^{just-tuned} \mapsto \mathbb{N}_+ :$

$$G_{Eul}(\mu , \nu) := G_{Eul}(\frac{\nu}{\mu}) \tag{6.42}$$

As it has been observed by Mazzola, since the higher is the consonance the lower is the value of the gradus function, G_{Eul} should be more properly called gradus dissuavitatis: the total ordering $<_{Eul}$ of dissonant intervals induced by definition VI.20:

$$\begin{aligned} 1 <_{Eul} 5P <_{Eul} 4P <_{Eul} 3M =_{Eul} 6M <_{Eul} 3m \\ =_{Eul} 6m =_{Eul} 2M <_{Eul} 7m <_{Eul} \\ 7M <_{Eul} 2m <_{Eul} 5^{\circ} \end{aligned} \tag{6.43}$$

where I have adopted the standard musical notation for fourth and fifth perfect (P) as well as for major (M) , minor (m) and diminished (∘) intervals; this can been directly verified through the following expression of the mentioned Mathematica notebook of sectionA:

```
gradussuavitatis[n_Integer]:=
  1+Sum[  ( FactorInteger[n][[i]][[1]]-1)* FactorInteger[n][[i]][[2]]   ,
{i,1,
        Length[FactorInteger[n]]}]

gradussuavitatis[r_Rational]:=
  gradussuavitatis[
    Times[Numerator[r]*Denominator[r],
      Power[GCD[Numerator[r],Power[Denominator[r],2]]  , -1 ]]]
```

The next step in the history of the comprehension of *physical consonance* was made in 1877 by Hermann L. F. Helmhotz in his masterpiece [59] whose conclusions are condensed in the following passage:

"When two musical tones are sounded at the same time, their united sound is generally disturbed by beats of the upper partials, so that a greater or less part of the whole mass of sound is broken up into pulses of tone, and the joint effect is rough. These exceptional cases are cold Consonances. From the Part II: "On the interruptions of harmony", cap. 10: "Beats of the Upper Partial Tones", section "Order of Consonances with respect to Harmoniousness", pagg. 194-197 of [59]

"When two or more compound tones are sounded at the same time beats may arise from the combinational tones as well from the harmonic upper partials. In Chapter VII it was shown that the loudest combinational tones resulting from two generating tones is that corresponding to the difference of their pitch numbers, or the differential tone of the first order. It is this combinational tone, therefore, which is chiefly effective in producing beats" From the Part II: "On the interruptions of harmony", cap.11: "Beats due to combinational tones" pagg.197-198 of [59]

To explain Helmholtz's point of view it is essential to distinguish among *interference phenomena* prescribed by Linear Acoustics and the *nonlinear combinational effects* owed to violations of the axiom V.1

let us suppose to have two distinct sonore sources $i_1(t)$ and $i_2(t)$ suct that each one, from its own, would produce, if it was alone, respectively the sounds:

$$s_1(t) \; := \; \mathcal{R}_{ac}[i_1(t)] \; = \; a_1 e^{i\omega_1 t} \tag{6.44}$$

and

$$s_2(t) \; := \; \mathcal{R}_{ac}[i_2(t)] \; = \; a_2 e^{i\omega_2 t} \tag{6.45}$$

LINEAR REGIME Assuming the linearity of medium's acoustic response, i.e. the axiom V.1 it follows that:

$$\begin{aligned} s(t) \; &= \; \mathcal{R}_{ac}[i_1(t) + i_2(t)] \; = = \mathcal{R}_{ac}[i_1(t)] \; + \; \mathcal{R}_{ac}[i_2(t)] \\ &= \; s_1(t) + s_2(t) \; = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \end{aligned} \tag{6.46}$$

Posed:

$$\Delta\,\omega \; := \; \omega_2 \; - \; \omega_1 \tag{6.47}$$

and:

$$a_1 \; = \; |a_1| e^{i\phi_1} \tag{6.48}$$

$$a_2 \; = \; |a_2| e^{i\phi_2} \tag{6.49}$$

it follows that:

$$s(t) \; = \; a(t)\, e^{i\omega_1 t} \tag{6.50}$$

where:

$$|a(t)| \; = \; \sqrt{|a_1|^2 \; + \; |a_2|^2 \; + \; 2|a_1|\,|a_2| \; \cos(\phi_1 - \phi_2 - \Delta\,\omega))} \tag{6.51}$$

and:

$$Arg(a(t)) \; = \; \arctan[\frac{|a_1| \sin(\phi_1) \; + \; |a_2| \sin(\phi_2 + \Delta\,\omega\, t)}{|a_1| \cos(\phi_1) \; + \; |a_2| \cos(\phi_2 + \Delta\,\omega\, t)}] \tag{6.52}$$

The qualitative behavior prescribed by equation eq.6.50, eq.6.51 and eq.6.52 is with some respect similar to that of an *amplitude modulation* in which an harmonic *carrier signal* of pulsation ω_1 is modulated by an harmonic *envelope signal* of pulsation $\Delta\omega$.

Though similar, the signal described by eq.6.50, eq.6.51 and eq.6.52 is not an *amplitude modulating signal* that would have not only spectral component $\omega_1 + \Delta\omega$ but also spectral component $\omega_1 - \Delta\omega$.

NON LINEAR REGIME

Since, usually the acoustical response of the medium may be expressed as the sum of a linear and nonlinear part, one has that:

$$\begin{aligned} s(t) \;=\; \mathcal{R}_{ac}[i_1(t)+i_2(t)] \;=\; \mathcal{R}^{L}_{ac}[i_1(t)+i_2(t)] \;+\; \mathcal{R}^{NL}_{ac}[i_1(t)+i_2(t)] \\ = \mathcal{R}^{L}_{ac}[i_1(t)] \;+\; \mathcal{R}^{L}_{ac}[i_2(t)] \;+\; \mathcal{R}^{NL}_{ac}[i_1(t)+i_2(t)] \end{aligned} \tag{6.53}$$

It may be worth observing that, in the nonlinear regime, one has to give up the complex representation of harmonic motion since:

$$\Re\mathcal{R}^{NL}_{ac}[\tilde{i}(t)] \;\neq\; \mathcal{R}^{NL}_{ac}[i(t)] \tag{6.54}$$

Combinational tones arise when the non linear part of the acoustical response function is given by the product of the two input signals.

One has that

$$\begin{aligned} \mathcal{R}^{NL}_{ac}[i_1(t)+i_2(t)] \;=\; \mathcal{R}^{NL}_{ac}[i_1(t)] \cdot \mathcal{R}^{NL}_{ac}[i_2(t)] \\ =\; s_1(t) \cdot s_2(t) \;= a_1\cos(\omega_1 t+\phi_1) \cdot a_2\cos(\omega_2 t+\phi_2) \\ =\; \frac{a_1 a_2}{2}\cos[(\omega_1+\omega_2)+(\phi_1+\phi_2)] \;+\; \frac{a_1 a_2}{2}\cos[(\omega_1-\omega_2)+(\phi_1-\phi_2)] \end{aligned} \tag{6.55}$$

Consequentially the net acoustic response consists not only of the two notes with pulsations ω_1 and ω_2 but also of other 2 notes with pulsations $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$.

HORIZONTAL RULES IN SCORES The theory of dissonance resolution inside a fixed tonality and the theory of modulation ruling how to connect different tonalities

VII. VERTICAL RULES REQUIRE THE COMMENSURABILITY OF NOTES

Given two sounds:

$$s_1(t) \; = \; \sum_{n=-\infty}^{+\infty} a_n^{(1)} e^{in\omega_1 t} \tag{7.1}$$

$$s_2(t) \; = \; \sum_{n=-\infty}^{+\infty} a_n^{(2)} e^{in\omega_2 t} \tag{7.2}$$

let us formalize the original Galilei's viewpoint mixed with a bit of Helmotz, and let us consider the etymology of the term consonance: we are led to select the harmonics of the two sounds that are commensurable.

DEFINITION VII.1

PHYSICAL INDEX OF CONSONANCE AMONG s_1 and s_2:

$$I[s_1\, ,\, s_2] \; := \; \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \delta_{Kronecker}(n\omega_1 + m\omega_2)\, (a_n^{(1)} + a_m^{(2)}) \tag{7.3}$$

where:

$$\delta_{Kronecker}(\omega) \; := \; \begin{cases} 1, \text{ if } \omega \; = \; 0 \; ; \\ 0, \text{ otherwise;} \end{cases} \tag{7.4}$$

DEFINITION VII.2

ω_1 AND ω_2 ARE COMMENSURABLE ($\omega_1 \; \sim_{\mathbb{Q}} \; \omega_2$):

$$\frac{\omega_1}{\omega_2} \; \in \; \mathbb{Q} \tag{7.5}$$

Theorem VII.1

CONSONANCE REQUIRES THE COMMENSURABILITY OF 2 NOTES:

$$\omega_1 \; \nsim_{\mathbb{Q}} \; \omega_2 \; \Rightarrow \; I[s_1, s_2] \; = \; 0 \tag{7.6}$$

More generally, given N sounds $s_1 \; \cdots \; s_N$:

$$s_i(t) = \sum_{n=-\infty}^{+\infty} a_n^{(i)} e^{in\omega_i t} \; i = 1, \cdots N \tag{7.7}$$

DEFINITION VII.3

PHYSICAL INDEX OF CONSONANCE AMONG $s_1 \cdots s_N$:

$$I[s_1 , \cdots s_N] \; := \; \sum_{n_1, \cdots n_N = -\infty}^{+\infty} \delta_{Kronecker} (\sum_{i=1}^{N} \omega_i n_i) \, (\sum_{i=1}^{N} a_n^{(i)}) \tag{7.8}$$

DEFINITION VII.4

$\omega_1 \cdots \omega_N$ ARE COMMENSURABLE ($\omega_1 \sim_{\mathbb{Q}} \cdots \sim_{\mathbb{Q}} \omega_N$):

$$\exists n_1 \cdots n_N \in \mathbb{Z} \; : \; \sum_{i=1}^{N} \omega_i n_i \; = \; 0 \tag{7.9}$$

Theorem VII.2

CONSONANCE REQUIRES THE COMMENSURABILITY OF N NOTES:

$$\omega_1 \nsim_{\mathbb{Q}} \cdots \nsim_{\mathbb{Q}} \omega_N \; \Rightarrow \; I[s_1, \cdots , s_N] \; = \; 0 \tag{7.10}$$

DEFINITION VII.5

MUSICAL INSTRUMENT:

$$\bar{a} \; = \; \{a_n\}_{n \in \mathbb{Z}} \; \in \; \mathbb{C}^{\infty} \; : \; \sum_{n=-\infty}^{\infty} a_n a_{-n} \; < \; +\infty \tag{7.11}$$

Given N notes $\omega_1 \cdots \omega_N$ and a musical instrument $\bar{a}$:

DEFINITION VII.6

PHYSICAL INDEX OF CONSONANCE OF $\omega_1 \cdots \omega_N$ W.R.T. $\bar{a}$:

$$I(\omega_1 \cdots \omega_N \, | \, \bar{a}) \; := \; I(\sum_{n=-\infty}^{+\infty} a_n e^{i n \omega_1 t} , \cdots , \sum_{n=-\infty}^{+\infty} a_n e^{i n \omega_n t}) \tag{7.12}$$

Example VII.1

PHYSICAL CONSONANCE OF NOTES W.R.T. A OF PURE OSCILLATOR:

A pure oscillator is, by definition, a musical instrument of the form:

$$a_n \; := \; a\,\delta_{n,k} \tag{7.13}$$

Then:

$$I(\omega_1, \omega_2 \,|\, a_n) \; = \; \begin{cases} 2\,a, & \text{if } \omega_1 \, = \, \omega_2 \\ 0, & \text{otherwise.} \end{cases} \tag{7.14}$$

Example VII.2

PHYSICAL CONSONANCE OF NOTES W.R.T. AN IDEAL MUSICAL INSTRUMENT

The exampleVII.1 shows that, as to the possibility of generating consonants sounds, a pure oscillator is the most inefficient musical instrument: the only consonant notes it can product are unisonous ones.

Looking at definitionVII.6 it should be clear why, contrary, an ideal instrument, i.e. the most efficient musical instrument as to production of consonance, is one in which the amplitudes of higher harmonics decades in the slowest possible way; we are then led to define an ideal musical instrument by the condition:

$$a_n \; := \; \frac{a}{n} \tag{7.15}$$

Then:

$$I(\omega_1, \omega_2 \,|\, a_n) \; = \; \begin{cases} +\infty, & \text{if } \omega_1 \sim_{\mathbb{Q}} \omega_2 \; ; \\ 0, & \text{otherwise.} \end{cases} \tag{7.16}$$

We would like to advise the reader that the wrong expression $\frac{n+m}{nm}$ given in the 6^{th} chapter of [23] for the physical index of consonance among two commensurable notes with ratio of frequencies $\frac{m}{n}$ with $gcd(m,n) = 1$ is based on the erroneous procedure of defining I up to an infinite constant (the harmonic serie).

VIII. HORIZONTAL RULES REQUIRE THE INCOMMENSURABILITY OF NOTES

The theory of dissonance resolution inside a fixed tonality and the theory of modulation ruling how to connect different tonalities are based on the same central structural element: the *scales at fixed interval*:

given two positive real numbers ω_1 , $R \in \mathbb{R}_+$:

DEFINITION VIII.1

SCALE OF ω_1 AT FIXED INTERVAL R:

$scale(\omega_1 \, , \, R) \; := \; \{\omega_n\}_{n=1}^{N}$:

$$\omega_n \quad = \quad R\omega_{n-1} \, mod(2^{\mathbb{N}}) \;\; n = 1, \cdots, N \tag{8.1}$$

$$N \quad := \quad \begin{cases} \min\{n \in \mathbb{N} \, : \, \omega_N \, = \, \omega_1 \, mod(2^{\mathbb{N}})\}, & \text{if } \min\{\cdots\} \text{ exists} \\ +\infty, & \text{otherwise} \end{cases} \tag{8.2}$$

DEFINITION VIII.2

$scale(\omega_1 \, , \, R)$ IS FINITE

$$N \; < \; +\infty \tag{8.3}$$

We have that:

Theorem VIII.1

FINITENESS VERSUS THE IRRATIONALITY OF THE FIXED INTERVAL:

$$scale(\omega_1 \, , \, R) \text{ is finite } \; \Leftrightarrow \; R \notin \mathbb{Q} \tag{8.4}$$

PROOF:

The thesis follows immediately by the fact that:

$$\sqrt[n]{2} \, \notin \, \mathbb{Q} \;\; \forall n \in \mathbb{N} \, : \, n \geq 2 \tag{8.5}$$

■

Corollary VIII.1

FINITENESS VERSUS THE INCOMMENSURABILITY OF NOTES:

$$scale(\omega_1\,,\,R) \text{ is finite } \Leftrightarrow \; \omega_i \nsim_{\mathbb{Q}} \omega_j \, i \neq j < N \tag{8.6}$$

PROOF:

It is sufficient to verify that $\sim_{\mathbb{Q}}$ is an equivalence relation and hence, in particular, is transitive ■

Let us now observe that the notes in a scale at fixed interval $scale(\omega_1\,,\,R)$ don't occur in order of increasing frequency; given a string or a sequence of real numbers $\{a_n\}$

DEFINITION VIII.3

ORDERING OF $\{a_n\}$:

$$ord[\{a_n\}] \;=\; \{\tilde{a}_n\} \;:=\; permutation(\{a_n\}) \;:\; i < j \;\Rightarrow\; \tilde{a}_i \;<\; \tilde{a}_j \tag{8.7}$$

Example VIII.1

PYTAGORIC SCALE OF ω_1:

The pytagoric scale of a note is constructed following the *natural cycle of fifths*, i.e. the cycle of fifths assuming as fifth interval the one specified by the second harmonic of ω_1 equal to 3:

$$pytagoric(\omega_1) \;:=\; scale(\omega_1\,,\,3) \tag{8.8}$$

By theorem VIII.1 it immediately follows that pytagoric scales are always infinite.

Remark VIII.1

ORDERING THE PYTAGORIC SCALES GLOBALLY OR AT FIFTH CYCLES BLOCKS

Let us observe that:

$$\cdot_{n \in \mathbb{N}} ord(scale(\nu_{ref}, 3)_{n,n+11}) \;\neq\; ord(scale(\nu_{ref}, 3)) \tag{8.9}$$

a fact, this one, we will see more concretelly in the section VIII.

Example VIII.2

N-EQUALLY-TEMPERED SCALES OF ω_1:

The N-equally-tempered scale of ω_1 is constructed following the*N-equally-tempered cycle of fifths*, i.e. the cycle of fifth adopting as basic fifth interval a value different from the one specified by Nature through the third harmonic, chosen in order of obtaining the periodicity of the cycle with period N.

Indeed, defining:

$$tempered_{eq}(\omega_1, N) \ := \ scale(\omega_1, 2^{\frac{7}{N}}) \tag{8.10}$$

by theorem VIII.1 it immediately follows that $tempered(\omega_1, N)$ is a finite scale of N notes.

We will now apply these considerations to show how the incommensurability of notes is required both by the theory of physical dissonance's resolution inside a fixed tonality and by the other ingredient horizontal rules in scores are made of, namely the theory of modulation between different tonalities.

At this purpose its is necessary, first of all, to introduce a rigorous mathematical formalization of both Classical Harmony and some generalizations of its.

Such an objective was pursued by the decennial work of the Guerino Mazzola's research's group about *Mathematical Music Theory*, [42], [9], [43].

Some preliminary consideration is required as to the functorial formalism in which the whole matter has been finally recasted:

the construction of the abstract mathematical machinery of *forms* and *denotators* has been justified by Mazzola through the following argumentations:

1. the temporally evolving nature of the global encyclopedic space of human knowledge (called by Mazzola *encyclospace* with an explicit referring to the assumption of its informatic storing in a virtual space such as a database embedded in Internet) possessing a topological structure endowed in the inter-relational links among distinct concepts (cfr. in particular the 5^{th} chapter "Navigation" and the 9^{th} chapter "Yoneda Perspectives" of [9])

2. the requirement to mirror such a structure into the structure of the software concretelly computing the mathematical concepts of encyclospace

While the latter argumentation, based on the well known role played by Category Theory in Theoretical Computer Science [60], [61], is clear corresponding to the concrete architecture

of the program RUBATO, the former argumentation requires some further analysis:

while in [43] Mazzola explicitely states that the existence and size of *form semiotics* is indeed an essential problem, observing that all the concretelly appearing forms of Mathematical Music Theory, being regular, can be obtained from *simple forms* through transfinite recursion, the 9^{th} chapter "Yoneda Perspectives" of [9] (more or less explicitely) states that the passage to the functorial formalism allows a consistent mathematical adoption of circular definitions.

Mazzola's claim is, with this regard, trivially false:

the passage from the set-theoretic foundations to the topos-theoretic foundations of Mathematics, corresponding to a passage from the Zermelo-Fraenkel formal system augmented with the Axiom of Choice, ZFC from here and beyond, to a formal system equiconsistent with the weaker Restricted Zermelo formal system augmented with the Axiom of Choice, RZC from here and beyond, doesn't drop the Foundation Axiom banning any kind of circularity (cfr. the 10^{th} section "Topos-Theoretic and Set-Theoretic Foundations" of the 6^{th} chapter "Topoi and Logic" of [10]):

the proof of the equiconsistence among the Topos- Theoretic Foundation of Mathematics and the set-theoretical foundation furnished by RZC, consists in:

1. showing that from each model $\mathcal{S}$ of RZC one can construct a well-pointed topos $\mathcal{E}$ (more precisely a model of the first-order-theory WPT of well-pointed topoi) with choice and natural number object
2. showing, conversely, that for each such well-pointed topos $\mathcal{E}$ one can construct a model of RZC.

The first part of the equi-consistence's proof constructing a well-pointed topos $\mathcal{E}$ from a model $\mathcal{S}$ of set-theory consist simply into the construction of the category of all sets of $\mathcal{S}$.

The second part of the equi-consistence's proof is, instead, more complex.

My purpose, with this regard, is to remark how, in such a construction of a model $\mathcal{S}$ of RCZ from a well-pointed topos with natural numbers object and choice, the Vicious Circle Principle (adopting the old terminology by Russell and Poincaré concerning the exorcism of Russell's paradox for whose discussion I demand to [62] and to the section 4.3 "The Set-Theoretical Hierarchy" of [63]) expressed by the following one among RZC's axioms:

AXIOM VIII.1

AXIOM OF FOUNDATION

$$(\exists y)(y \in x) \ \rightarrow \ (\exists y)[y \in x \ \wedge \ (\forall z \in y)(z \notin x)]$$

and stating that any nonempty set has a minimal element with respect to the set-theoretic membership relation $\in$ is granted by the Axiom of Well-Founded Up Trees contained into the axiomatic definition of a tree in (the Mitchell-Benabou language of) an *elementary topos* $\mathcal{E}$ stating that any non-empty subtree of a tree has a maximal element.

DEFINITION VIII.4

TREE ON $\mathcal{E}$

an object T of $\mathcal{E}$ endowed with a binary relation $R \mapsto T \times T$ such that:

Poset: R is a partial-order relation on T (and will be denoted from this reason as $\leq$ from here and beyond

Root:

$$\exists\, 0 \in T \ : \ (0 \leq t \ \forall t \in T) \tag{8.11}$$

Tree Property:

$$\downarrow t \text{ is linearly ordered by } \leq \ \forall t \in T \tag{8.12}$$

Well-founded down:

$$\forall S \subseteq T \ : \ S \neq \emptyset \, , \exists y \in S \ : \ y > z \ \Rightarrow \ z \notin S \tag{8.13}$$

(i.e. y is minimal in S w.r.t. $\leq$)

Well-founded up:

$$\forall S \subseteq T \ : \ (S \neq \emptyset \ \Rightarrow \ \exists w \in S \ : \ Not(z > w \ : \forall z \in S) \tag{8.14}$$

(i.e. w is maximal in S w.r.t. $\leq$)

rigid

$$\alpha : T \mapsto T \ \text{ automorfism of T } \alpha \, = \, \mathbb{I} \tag{8.15}$$

where $\mathbb{I}$ denotes the identity map on T while an automorphism of T is defined as a *leq* preserving bijection of T

and where:

DEFINITION VIII.5

DOWNWARD CLOSURE OF $t \in T$:

$$\downarrow t \;:=\; \{x \,:\, x \,\leq t\} \tag{8.16}$$

Informally and intuitivelly speaking, the key point consists in representing a set x as a graph in the following way: x is the root of the tree;on the first level are the members of x , joined to x through a directed edge. Next above there are the members of the members y of x, each one joined to its y though a directed edge.

Attempts to build "anti-foundational" axiomatizations of Mathematics giving up Russell's Vicious Circle Principle (that should more properly been called the Principle of Censorship of Vicious Circles) can only arise whether the constraint that the above graph doesn't contains cycles, and is conseguentially a tree, is removed.

One possibility is given by the theory of hyper-sets, based on formal system ZFA obtained by ZFC replacing the Axiom of Foundation with a suitable Axiom of Anti-foundation.

Hypersets may be reached in many different ways;

Following the 2^{th} section "Ensembles non-bien fondés " of [62] and the chapter 2 "Background on set theory" an the chapter 10 " Graphs" of [64] I will adhere to Aczel's formulations.

Let us assume a *proper class* $\mathcal{U}$ of ur-elements that are not sets, are not classes and have no members; given a set s:

DEFINITION VIII.6

s IS TRANSITIVE:

$$x \,\in\, s \;\Rightarrow\; x \in \mathcal{P}(s) \tag{8.17}$$

DEFINITION VIII.7

TRANSITIVE CLOSURE OF s:

$$TC(s) \;:=\; \bigcup \{a \,,\, \bigcup a \,,\, \bigcup \bigcup \, a \,,\, \cdots \} \tag{8.18}$$

DEFINITION VIII.8

SUPPORT OF s:

$$support(s) \ := \ TC(s) \bigcap \mathcal{U} \tag{8.19}$$

DEFINITION VIII.9

s IS PURE:

$$support(s) \ = \ \emptyset \tag{8.20}$$

Given an $A \subseteq \mathcal{U}$:

$$V_{afa}(A) \ := \ \{a \ : \ a \text{ is a set } \ and \, support(a) \ \subseteq \ A\} \tag{8.21}$$

DEFINITION VIII.10

CLASS OF THE PURE SETS:

$$V_{afa} \ = \ V_{afa}(\emptyset) \tag{8.22}$$

Let us now define in a rigorous way the notion of *graph*:

DEFINITION VIII.11

GRAPH:

a couple $\mathbf{G} \ := \ (G \, , \rightarrow_G)$ such that G is a set and $\rightarrow_G$ is a binary relation over G.

I will denote the proper class of all graphs by GRAPHS.

given $\mathbf{G} = (G \, , \rightarrow_G) \in GRAPHS$

DEFINITION VIII.12

d is DECORATION OF $\mathbf{G}$ $(\, d = D(\mathbf{G}) \,)$

if d is a function $d \ : \ \mathbf{G} \ \rightarrow \ V_{afa}$ such that:

$$d(a) \ = \ \{d(b) \ : \ a \ \rightarrow_G b\} \ \ \forall a, b \in \mathbf{G} \tag{8.23}$$

One can than introduce the following:

AXIOM VIII.2

AXIOM OF ANTI-FOUNDATION

$$\forall\, g \,\in\, GRAPHS\, \exists!\, D(g) \tag{8.24}$$

Let us now consider some formal system:

DEFINITION VIII.13

FORMAL SYSTEM ZFC^{-}

the formal system with axioms:

URELEMENTS

$$(\forall p)(\forall q)[\mathcal{U}(p) \;\rightarrow\; \neg(q \in p)]$$

EXTENSIONALITY

$$(\forall a)(\forall b)[(\forall p)[p \in a \;\leftrightarrow\; p \in b) \;\rightarrow\; a \;=\; b]$$

PAIRING

$$(\forall p)(\forall q)(\exists a)[p \in a \wedge q \in a]$$

UNION

$$(\forall a)(\exists b)(\forall c \in a)(\forall p \in c)p \in b$$

POWER SET

$$(\forall a)(\exists b)(\forall c)[c \subseteq a \;\rightarrow\; c \in b]$$

INFINITY

$$(\exists a)[\emptyset \in a \wedge (\forall b)[b \in a \;\rightarrow\; (\exists c \in a)c = b \cup \{b\}]]$$

COLLECTION

$$(\forall a)(\forall p \in a)(\exists q)\varphi(a, p, q) \;\rightarrow\; (\exists b)(\forall p \in a)(\exists q \in b)\varphi(a, p, q)$$

SEPARATION

$$(\forall a)(\exists b)(\forall p)[p \in b \;\leftrightarrow\; p \in a \wedge \Phi(p, a)$$

CHOICE

$$(\forall a)(\exists r)[\text{r is a well-order of a}]$$

STRONG PLENITUDE

$$(\forall a)(\forall b)[\mathcal{U}(new(a,b)) \ \wedge \ new(a,b) \notin b \ \wedge \ (\forall c \neq b)[new(a,b) \neq new(c,b)]] \tag{8.25}$$

DEFINITION VIII.14

FORMAL SYSTEM ZFC:

$$\text{ZFC} = ZFC^{-} \quad + \quad \text{AXIOM VIII.1}$$

DEFINITION VIII.15

FORMAL SYSTEM ZFA:

$$\text{ZFA} = ZFC^{-} \quad + \quad \text{AXIOM VIII.2}$$

The replacement of the axiomVIII.1 axiomVIII.2 clearly affects the way by which the comparison of sets is defined so that the Axiom of Extensionality has to be looked at in a rather different way as it was for ZFC. As to our considerations, it will be sufficient to know that ZFA results to be a (consistent ?) formal system about which I demand to [64] getting rid of Russell's paradox in a completelly non-orthodox way.

What it is important here to remark is that ZFA is not , of course, equiconsistent with the topos-theoretic foundation of Mathematics (first order theory of a Well-pointed Topos) and consequentially with RZC.

Whether Mazzola correctly claims that the passage to the Topos-theoretical formalization of Mathematics may be seen, from a philosophical perspective, as a kind of "behavioural revolution" according to which the identity of a mathematical concept is completely characterized by its properties as to inter-relation with other mathematical concepts, his claim that a such a passage contains the abrogation of the Vicious Circle Principle is , consequentially, completely false.

The claim that the so called "Yoneda Perspective" gives some kind of conceptual consistence to postmodern intellectual hoaxes trying to give an impression of conceptual consistence to circularities through semiological arguments or whatsoever (cfr. [65] and the 2^{th}

chapter "Dizionario versus Enciclopedia" of [15]) is consequentially an intellectual hoax by itself.

One could, indeed, use the equiconsistence of RZC and the first-order theory of well-point topoi to obtain a topos-theoretic version of ZFA. But such an anti-foundational formal system would be completelly a different thing with respect to the Lawvere's topos-theoretic foundation of Mathematics where, exactly as in the set theoretic foundation, circularity is banned.

Since Mazzola's further constructions have as a ground the formalization of the American school pioneered by Milton Babbit, the necessary step consists in comparing the information-theoretic language of [18] I will adopt and some of the basic notions of the american tradition as codified by Allen Forte in [66].

Let us consider the scale $tempered(C_2, 12)$;

imponing octave-periodicity it results that the musical alphabet may be represented as the set of the residue classes modulo twelve $\mathbb{Z}_{12}$, endowed with its algebraic structure of a ring w.r.t. to the operations $+_{12}$ and $-_{12}$.

Let us introduce the following maps:

DEFINITION VIII.16

TRANSLATION OF $y \in \mathbb{Z}_{12}$:

the map $T_y : \mathbb{Z}_{12} \mapsto \mathbb{Z}_{12}$ such that:

$$T_y(x) := x +_{12} y \;\; \forall x \in \mathbb{Z}_{12} \tag{8.26}$$

and:

DEFINITION VIII.17

INVERSION OPERATOR:

the map $Inv : \mathbb{Z}_{12} \mapsto \mathbb{Z}_{12}$ such that:

$$Inv(x) := [0]_{12} -_{12} x \tag{8.27}$$

It is curious to notice at this point how a non-prime equal-tempering was coherent with the classical viewpoint on esthetic consonance, summarized in axiom VI.1, giving rise to simpler, i.e. non co-prime ratios among the notes of physically consonant intervals; from the

viewpoint of Atonal Music a prime tempering would have been, contrary, more reasonable: since 12 is not prime the ring $\mathbb{Z}_{12}$ is not a (Galois) field and hence multiplication $\times_{12}$ and division $\div_{12}$ among *pitch class sets* cannot be defined so that one cannot implement the other two symmetry transformation that could, instead, be implemented on $\mathbb{Z}_{11}$ and $\mathbb{Z}_{13}$.

Of course the musical alphabet $\mathbb{Z}_{12}$ as well as the basic symmetry transformation are easily implemented by the following Mathematica expressions from section A:

```
referencenote= 132;

(*** the  following instructions implements the musical alphabet
Z-12 ***)

letter[n_]:=Mod[n,12]

alphabet=Table[letter[n],{n,0,11}];

FROMletterTOnote[n_] :=
referencenote*Power[2,Power[letter[n],Power[12,-1]]]
```

I will adopt from here and beyond also the musical notation:

$$C := [0]_{12} \quad (8.28)$$

$$C^{\sharp} := [1]_{12} \quad (8.29)$$

$$D := [2]_{12} \quad (8.30)$$

$$D^{\sharp} := [3]_{12} \quad (8.31)$$

$$E := [4]_{12} \quad (8.32)$$

$$F := [5]_{12} \quad (8.33)$$

$$F^{\sharp} := [6]_{12} \quad (8.34)$$

$$G := [7]_{12} \quad (8.35)$$

$$G^{\sharp} := [8]_{12} \quad (8.36)$$

$$A := [9]_{12} \quad (8.37)$$

$$A^{\sharp} := [10]_{12} \quad (8.38)$$

$$B := [11]_{12} \quad (8.39)$$

$$(8.40)$$

Let us then consider the set $\mathbb{Z}_{12}^{\star}$ of all the finite strings (that I will also call words from here and beyond) over $\mathbb{Z}_{12}$, each element of which is nothing but an *ordered pitch class set* in Forte's terminology.

Once more we can algorithmically implement the introduced notions through the following Mathematica expression of the notebook reported in sectionA:

```
FROMletterTOnote[n_] :=
referencenote*Power[2,Power[letter[n],Power[12,-1]]]


FROMwordTOscale[word_]:=
  Table[FROMletterTOnote[Part[word,i]],{i,1,Length[word]}]


(*** the following instructions introduced the notes without the
Mod-12  constraint ***)


c[1]=FROMletterTOnote[0];
```

```
c[n_]:=2*c[n-1]

c\[Sharp][1]=FROMletterTOnote[1];

c\[Sharp][n_]:=2*c\[Sharp][n-1]

d[1]=FROMletterTOnote[2];

d[n_]:=2*d[n-1]

d\[Sharp][1]=FROMletterTOnote[3];

d\[Sharp][n_]:=2*d\[Sharp][n-1]

e[1]=FROMletterTOnote[4];

e[n_]:=2*e[n-1]

f[1]=FROMletterTOFROMletterTOnote[5];

f[n_]:=2*f[n-1]

f\[Sharp][1]=FROMletterTOnote[6];

f\[Sharp][n_]:=2*f\[Sharp][n-1]

g[1]=FROMletterTOnote[7];

g[n_]:=2*g[n-1]

g\[Sharp][1]=FROMletterTOnote[8];
```

```
g\[Sharp][n_]:=2*g\[Sharp][n-1]

a[1]=FROMletterTOnote[9];

a[n_]:=2*a[n-1]

a\[Sharp][1]=FROMletterTOnote[10];

a\[Sharp][n_]:=2*a\[Sharp][n-1]

b[1]=FROMletterTOnote[11];

b[n_]:=2*b[n-1]
```

where the recursive definition of $x[n]$ $x = C, \cdots, B$ $n > 1$ has being introduced in order to allow concretelly to play also higher octaves.

With this regard, if a a monodic piece is represented as a list each element of which is itself a list of the form $\{$ *note* , *duration*$\}$ the involved code is:

```
(*** a monodic piece is a list each element of which is itself a list of the
    form {note ,duration}   ***)

playmonodic[piece_]:=
  Do[Play[Sin[piece[[i]][[1]] *2*\[Pi]*t],{t,0,piece[[i]][[2]]}] , {i,1,
      Length[piece]}]
```

with the durations implemented as:

```
(*** insert the "referencetime" ***)

referencetime=4;
```

```
semibreve=1*referencetime;

minim= (1/2)*referencetime;

crotchet= (1/4)*referencetime;

quaver=(1/8)*referencetime;

semiquaver=(1/16)*referencetime;

demisemiquaver=(1/32)*referencetime;

hemidemisemiquaver=(1/64)*referencetime;
```

and with the following instructions allowing to associate (and to eliminate) a fixed crothet duration:

```
FROMscaleTOpiece[scale_]:=Table[{Part[scale,i],crotchet},{i,1,Length[scale]}]

FROMpieceTOscale[piece_]:=Table[Part[piece,i,1],{i,1,Length[piece]}]
```

Given a word $\vec{x} \in \mathbb{Z}_{12}^{\star} - \bigcup_{k=1}^{4} \mathbb{Z}_{12}^{k}$ and an integer $i \in \{1, \cdots, |\vec{x}|\}$:

DEFINITION VIII.18

MODE OF $\vec{x}$ of i^{th} DEGREE :

$$mode(\vec{x}\,,\,i) \;:=\; \mathcal{S}_{cycl}^{i}(\vec{x}) \tag{8.41}$$

where $\mathcal{S}_{cycl}$ denotes the operator of cyclic shift.

Given furthermore an integer $n \in \mathbb{N}$

DEFINITION VIII.19

CHORD OF $\vec{x}$ of i^{th} DEGREE AT LEVEL n :

$$chord(\vec{x}\,,\,i\,,\,n) \;:= \cdot_{j=0}^{n+3} mode(\vec{x}\,,\,i)_{2j+1} \tag{8.42}$$

where $\cdot$ denotes the concatenation operator while x_j denotes the j^{th} letter of the word $\vec{x}$.

Let us observe,at this point, that exists a certain maximum level $maxlevel(\vec{x})$ such that $chord(\vec{x}\,,\,i\,,\,n)$ for $n > maxlevel(\vec{x})$ simply adds notes already contained in the chord.

A simple reasoning allows to infer that:

$$maxlevel(\vec{x}) \;=\; 1 + Int(\frac{|\vec{x}|-5}{2}) - \frac{(-1)^{|\vec{x}|}-1}{2}\dot{I}nt(\frac{|\vec{x}|}{2}) \tag{8.43}$$

Given our alphabet $\Sigma := \mathbb{Z}_{12}$, a word $\vec{x} \in \Sigma^{\star}$ and a map $g \,:\, \Sigma \rightarrow \Sigma$:

DEFINITION VIII.20

MAP INDUCED BY g ON WORDS:

the map $\hat{g} \,:\, \Sigma^{\star} \,\rightarrow\, \Sigma^{\star}$:

$$\hat{g}(\vec{x}) \;=\; \cdot_{i=1}^{|\vec{x}|}\, g(x_i) \tag{8.44}$$

Applying , in particular, the definitionVIII.20 to the maps of definition VIII.16 and definition VIII.17 one obtains the following

DEFINITION VIII.21

TRANSLATION OF $y \in \mathbb{Z}_{12}$ ON WORDS:

the map $\hat{T}_y \,:\, \mathbb{Z}_{12}^{\star} \,\mapsto\, \mathbb{Z}_{12}^{\star}$

DEFINITION VIII.22

INVERSION OPERATOR ON WORDS;

the map $\hat{Inv} \,:\, \mathbb{Z}_{12}^{\star} \,\mapsto\, \mathbb{Z}_{12}^{\star}$

These operators naturally induce the following equivalence relations:

DEFINITION VIII.23

$\vec{x}_1\,,\,\vec{x}_2 \,\in\, \mathbb{Z}_{12}^{\star}$ ARE TRANSLATIONALLY-EQUIVALENT ($\vec{x}_1 \,\sim_T\, \vec{x}_2$):

$$\exists y \in \mathbb{Z}_{12} \;:\; \vec{x}_2 \,= T_y \vec{x}_1 \tag{8.45}$$

DEFINITION VIII.24

$\vec{x}_1 \, , \, \vec{x}_2 \, \in \, \mathbb{Z}_{12}^{\star}$ ARE INVERSIONALLY-EQUIVALENT ($\vec{x}_1 \, \sim_{Inv} \, \vec{x}_2$):

$$\vec{x}_2 \, = Inv \vec{x}_1 \tag{8.46}$$

that may be managed through the following instructions from sectionA:

```
(*** the following instructions implement the two operation of translation
    and inversion at the basis of atonal music as well as the relative
    tests ***)

translation[word_,n_]:=Table[Mod[Part[word,i]+n,12],{i,1,Length[word]}]

translationequivalenceofwordsQ[w1_,w2_]:=
  Not[Equal[Table[Equal[w2,translation[w1,n]],{n,0,11}],Table[False,{12}]]]

inversion[word_]:=Table[Mod[12-Part[word,i],12],{i,1,Length[word]}]

inversionequivalenceofwordsQ[w1_,w2_]:=Equal[w1,inversion[w2]]

inversioninvarianceofawordQ[w_]:=Equal[w,inversion[w]]
```

Let us now analyze the musical intervals inside a word: instead of the complicated Forte's approach I will adopt a more intuitive definition of the *interval vector*:

given two musical letters $x, y \in \mathbb{Z}_{12}$:

DEFINITION VIII.25

DISTANCE BETWEEN x AND y :

$$d(x \, , \, y) \; := \; x \, -_{12} \, y \tag{8.47}$$

Given a vord $\vec{x} \in \mathbb{Z}_{12}^{\star}$:

DEFINITION VIII.26

INTERVAL VECTOR OF $\vec{x}$:

$$I(\vec{x}) \; := \; (d(x_2, x_1) \, , \, \cdots \, , d(x_{|\vec{x}|}, x_{|\vec{x}|-1})) \tag{8.48}$$

One has obviously that:

Theorem VIII.2

THE INTERVAL VECTOR DETERMINES A TRANSLATIONAL EQUIVALENCE'S CLASS

$$\vec{x} \sim_T \vec{y} \Leftrightarrow I(\vec{x}) = I(\vec{y}) \tag{8.49}$$

PROOF:

Proof of the implication $\Rightarrow$ By hypothesis:

$$\exists z \in \mathbb{Z}_{12} : \vec{y} = T_z \vec{x} \tag{8.50}$$

One has that:

$$I(\vec{y}) = (d(x_2 +_{12} z, x_1 +_{12} z), \cdots, d(x_{|\vec{x}|} +_{12} z, x_{|\vec{x}|-1} +_{12} z)) = \\ = (d(x_2, x_1), \cdots, d(x_{|\vec{x}|}, x_{|\vec{x}|-1}) = I(\vec{x}) \tag{8.51}$$

Proof of the implication $\Leftarrow$ By hypothesis:

$$(d(x_2, x_1), \cdots, d(x_{|\vec{x}|}, x_{|\vec{x}|-1}) = (d(y_2, y_1), \cdots, d(y_{|\vec{x}|}, y_{|\vec{y}|-1}) \tag{8.52}$$

This implies that:

$$\exists z \in \mathbb{Z}_{12} : (d(y_2, y_1), \cdots, d(y_{|\vec{x}|}, y_{|\vec{y}|-1}) = \\ = (d(x_2 +_{12} z, x_1 +_{12} z), \cdots, d(x_{|\vec{x}|} +_{12} z, x_{|\vec{x}|-1} +_{12} z)) \tag{8.53}$$

implying that:

$$\vec{y} = T_z \vec{x} \tag{8.54}$$

■

Let us now fix a useful notation: words will be represented enclosed by $\{\cdots\}$ while interval vectors,and hence translationally-equivalences' classes, will be represented enclosed by $(\cdots)$.

So, for example, we will speak about the diatonic major scale $(2, 2, 1, 2, 2, 2, 1)$ referring to the translationally equivalence class:

$$(2, 2, 1, 2, 2, 2, 1) = [0, 2, 4, 5, 7, 9, 11]_T \tag{8.55}$$

Let us now introduce a particularly important subset of musical words. Let us consider, at this purpose, a generic finite (i.e. $card(\Sigma) < \infty$) alphabet Σ. We have clearly that:

$$
\begin{aligned}
card(\Sigma^n) &= (card(\Sigma))^n && (8.56)\\
card(\Sigma^\star) &= \aleph_0 && (8.57)\\
card(\Sigma^\infty) &= \aleph_1 && (8.58)
\end{aligned}
$$

Let us now introduce the following:

DEFINITION VIII.27

NONREPETITIVE WORDS OVER Σ OF LENGTH $n \in \mathbb{N}$

$$\Sigma^n_{NR} := \{\vec{x} \in \Sigma^n : x_i \neq x_j \ \forall i \neq j\} \tag{8.59}$$

DEFINITION VIII.28

NONREPETITIVE WORDS OVER Σ:

$$\Sigma^\star_{NR} := \{\vec{x} \in \Sigma^\star : x_i \neq x_j \ \forall i \neq j\} \tag{8.60}$$

Trivial combinatorial considerations allow to prove the following:

Theorem VIII.3

CARDINALITIES OF THE SETS OF NONREPETITIVE WORDS:

$$
\begin{aligned}
card(\Sigma^n_{NR}) &= \binom{card(\Sigma)}{n} && (8.61)\\
card(\Sigma^\star_{NR}) &= \sum_{k=1}^{card(\Sigma)} \binom{card(\Sigma)}{k} && (8.62)
\end{aligned}
$$

Applying theoremVIII.3 to our particular musical alphabet $\mathbb{Z}_{12}$ one obtains the following:

Corollary VIII.2

CARDINALITIES OF THE SET OF NONREPETITIVE MUSICAL WORDS

$$
\begin{aligned}
card(\mathbb{Z}^n_{12\ NR}) &= \binom{12}{n} && (8.63)\\
card(\mathbb{Z}^\star_{12\ NR}) &= \sum_{k=1}^{card(\Sigma)} \binom{12}{n} \ = \ 4095 && (8.64)
\end{aligned}
$$

Mazzola's strong generalization of the notion of tonality is based on the formalization of the 3-voices harmonization of the degrees of arbitrary nonrepetitive words of length equal to seven Σ_{NR}^{7}.

Gregorian Music [17] was based on the following strongly smaller subset of Σ_{NR}^{7}:

DEFINITION VIII.29

GREGORIAN-WORDS

$$\mathcal{S}_{greg} \; := \; \{ \, mode(\vec{x}, i) \, \vec{x} \in \Sigma_{NR}^{7} \; : \; I(\vec{x}) = (2,2,1,2,2,2,1) \, , \, i = 1 \; \cdots \; 7 \} \tag{8.65}$$

One has clearly that:

$$card(\mathcal{S}_{greg}) \; = \; 84 \tag{8.66}$$

since $\mathcal{S}_{greg}$ contains the 12 version starting from C to B, of the following modes:

$$I(s) \; = \; \begin{cases} (2,2,1,2,2,2,1), \; \text{ionian (major)} \\ (2,1,2,2,2,1,2), \; \text{dorian} \\ (2,1,2,2,2,1,2), \; \text{phrigian} \\ (2,2,2,1,2,2,1), \; \text{lydian} \\ (2,2,1,2,2,1,2), \; \text{myxolydian} \\ (2,1,2,2,1,2,2), \; \text{aeolian (minor)} \\ (1,2,2,1,2,2,2), \; \text{locrian} \end{cases} \tag{8.67}$$

i.e. all the elements of Σ_{NR}^{7} whose labels are made of 2 with the exception of two 1's at distance 4.

Classical Harmony, furthermore, is based on the following smaller subset of $\mathcal{S}_{greg}$:

DEFINITION VIII.30

CLASSICAL-WORDS

$$\mathcal{S}_{cl} \; := \; \{ [i]_{12} - ionian \, , \, [i]_{12} - aeolian \; , \; i \; = \; 1 \, , \, \cdots \, 12 \} \tag{8.68}$$

One has clearly that:

$$card(\mathcal{S}_{cl}) \; = \; 24 \tag{8.69}$$

[17] We will adopt here the standard denomination of gregorian modes demanding to [67] , [68] and to the fourth chapter "Le scale" of [29] for any further information concerning their functionalities in the catholic liturgy and their hystorical derivation from greek modes

Among the overwhelming majority of non-gregorian (and hence also non-classical) non-repetitive words worth of note are the harmonic minor scale (2,1,2,2,1,3,1) and its derived modes (locrian $\sharp 6$ $(1,2,2,1,3,1,2)$, augmented ionic $(2,2,1,3,1,2,1)$, minor lydian $7\flat$ $(2,1,3,1,2,1,2)$, myxolidian $\flat 2 / 6\flat$ $(1,3,1,2,1,2,2)$, lydian $\sharp 2$ $(3,1,2,1,2,2,1)$, super-locrian 7 diminuished $(1,2,1,2,2,1,3)$), the melodic minor scale $(2,1,2,2,2,2,1)$ and its derived modes (dorian $\flat 2$ $(1,2,2,2,2,1,2)$, augmented lydian $(2,2,2,2,1,2,1)$, dominating lydian $(2,2,2,1,2,1,2)$, myxolidian $\flat 6$ $(2,2,1,2,1,2,2)$, locrian $\sharp 2$ $(2,1,2,1,2,2,2)$, super-locrian $(1,2,1,2,2,2,2)$) as well as the 12-equally tempered approcimation of many ethnic scales cfr. the voices "Modi (Modalitá" and "Scale musicali antiche e moderne" of [69] and [70] [18] such as the major tzigan $(1,3,2,1,1,3,1)$ and minor tzigan scale $(2,1,3,1,1,3,1)$, the indian scale $(1,3,1,2,1,3,1)$, the indú scale $(2,2,1,2,1,2,2)$, the hungarian scale $(3,1,2,1,2,1,2)$ and the minor napoletan scale $(1,2,2,2,1,3,1)$.

We are now ready to introduce the following notions:

DEFINITION VIII.31

MAZZOLA TONALITY:

a couple $(\vec{x}\,,\, chord^{(3)})$ where:

1.
$$\vec{x} \,\in\, \Sigma_{NR}^{7} \tag{8.70}$$

2. $chord^{(3)} \;:\; \{1,2,3,4,5,6,7\} \;\mapsto\; \mathbb{Z}_{12}^{3}$ is the map associating to to the i^{th}-degree the chord:
$$chord^{(3)}(i) \;=\; chord(\vec{x}\,,\, i\,,\, 1) \tag{8.71}$$

Since a Mazzola-tonality is determined by its underlying non-repetitive eptatonic words, the set $\mathcal{T}_{Maz}$ of all the possible Mazzola-tonalities is bijective to Σ_{NR}^{7} , so that, clearly, one have

[18] It should be observed that it is only by 12-equally tempered approximation of the makam "Ahavah Rabbah" that it collapes to the fifth's mode of the harmonic minor scale. These brief and very schematic observations could be considered, anyway, as no more than a taste about the complexity of all the studies concerning Jewish Music, its identity, its esthetics (and consequentially, according to those insisting on the link music versus poietry , cfr. the subsection "Musica e Poesia" of the 2^{th} chapter "L'Occidente cristiano e l'idea di Musica" of [11]), its poetics [71]), the bi-directional flow of information with Christian music in its hystorical develop [72]. I demand to [73], [74], [75] and [76] for any further information

that:

$$card(\mathcal{T}_{Maz}) \;=\; 792 \tag{8.72}$$

Similarly, introduced the following notions:

DEFINITION VIII.32

GREGORIAN TONALITIES:

$$\mathcal{T}_{greg} \;:=\; \{(\vec{x}\,,\, chord^{(3)}) \;:\; \vec{x} \in \mathcal{S}_{greg}\} \tag{8.73}$$

DEFINITION VIII.33

CLASSICAL TONALITIES:

$$\mathcal{T}_{cl} \;:=\; \{(\vec{x}\,,\, chord^{(3)}) \;:\; \vec{x} \in \mathcal{S}_{cl}\} \tag{8.74}$$

the bijectivity of $\mathcal{S}_{greg}$ and $\mathcal{T}_{greg}$ and the bijectivity of $\mathcal{S}_{cl}$ and $\mathcal{T}_{cl}$ imply that:

$$card(\mathcal{T}_{greg}) \;=\; 84 \tag{8.75}$$

$$card(\mathcal{T}_{cl}) \;=\; 24 \tag{8.76}$$

Let us now observe that, as to the possibility of performing the harmonization of the degrees on the scale, there is no reason to restrict ourselves to Σ_{NR}^{7}; let us observe, with this regard [69], [70], that the popular music of many cultures adopts commonly the pentatonic scale $(2,2,3,2,4)$ and its modes, that the 12-equally tempered approximation of some ethnic scale such as the japanese Hira Joski $(1,4,1,4,2)$ or the japanese pelog $(1,4,1,4,2)$ are also pentatonic, that the blues scale $(3,2,1,1,3)$ is made of six notes and the many be-bop scales such as the bebop major $(2,2,1,2,1,1,2,1)$ or the be-bop dominant $(2,2,1,2,2,1,1,1)$ and the relative mode are made of 8 notes and so on.

Looking at Jazz Harmony [77], [78], [79], [80] furthermore, it appears natural to give up the restriction of considering only first-level chords.

One arrives, consequentially, to the following definition: given a word $\vec{x} \in \Sigma_{NR}^{\star} - \bigcup_{k=1}^{4} \Sigma_{NR}^{k}$ and a number $n \in \mathbb{N}_{+}$ such that $n \;<\; maxlevel(\vec{x})$

DEFINITION VIII.34

TONALITY OF THE WORD $\vec{x} \in \Sigma_{NR}^{\star} - \bigcup_{k=1}^{4} \Sigma_{NR}^{k}$ AT THE LEVEL n (n-TONALITY OF $\vec{x}$) :

$tonality(\vec{x}\,,\,n) \;:=\; (\vec{x}\,,\,chord^{(n)})$, where $chord^{(n)}\;:\;\{1,\cdots,|\vec{x}|\}\;\mapsto\;\mathbb{Z}_{12}^{\star}$ is the map such that:

$$chord^{(n)}(i) \;=\; chord(\vec{x}\,,\,i\,,\,n) \tag{8.77}$$

I will denote the set of all the n-tonalities by $\mathcal{T}_n$ while I will denote by $\mathcal{T}$ the set of all the tonalities at any level. One has clearly that:

$$card(\mathcal{T}) \;=\; \sum_{\vec{x}\in\Sigma_{NR}^{\star}-\bigcup_{k=1}^{4}\Sigma_{NR}^{k}} \; \sum_{n=1}^{maxlevel(\vec{x})} 1 \;=\; \sum_{k=5}^{12} card(\Sigma_{NR}^{k}) \sum_{n=1}^{1+Int(\frac{k-5}{2})-\frac{(-1)^{k}-1}{2}Int(\frac{k}{2})} 1$$
$$= \;10100 \tag{8.78}$$

Given an n-tonality $t \in \mathcal{T}_n$:

DEFINITION VIII.35

HARMONIC WORDS OF t:

$$\mathcal{HW}(t) \;:=\; (Range(chord^{(n)}(t)))^{\star} \tag{8.79}$$

Given two tonalities $t_1, t_2 \;\in\; \mathcal{T}$:

DEFINITION VIII.36

t_1 *AND* t_2 ARE TRANSLATIONALLY-EQUIVALENT ($t_1 \;\sim_T\; t_2$):

$$\exists z \in \mathbb{Z}_{12} \;:\; chord(t_1\; i) \;=\; T_z chord(t_2\; i) \;\;\forall i \tag{8.80}$$

Once again all these mathematical concepts are easily implemented trough the following expressions of the section A:

```
(*** mode[word, i])  gives the i - th mode of a word
on the musical alphabet   Z-12 ***)


mode[word_,i_]:=RotateLeft[word,i-1];
```

```
(***  chord[word,i,level] gives the chord of the i-th degree of a
word at a chosen level ***)

chord[word_,i_,level_]:= Table[Part[mode[word,i],

If[Mod[2n+1,Length[word]]==0,Length[word],Mod[2n+1,Length[word]]]],{n,0,
      1+level}]

tonality[word_,level_]:=Table[chord[word,i,level],{i,1,Length[word]}]

harmonicwords[t_,n_]:=Strings[t,n]

harmonicwordsupto[t_,n_]:=
  If[n==1,harmonicwords[t,1],
    Join[harmonicwordsupto[t,n-1],harmonicwords[t,n]]]

harmonicwordintermofdegrees[t_,listofdegrees_]:=
  Table[Part[t,Part[listofdegrees,i]],{i,1,Length[listofdegrees]}]

degreeofachordinatonality[w_,t_]:=Part[Flatten[Position[t,w]],1]

FROMharmonicwordTOphysicalchord[hw_]:=
  Table[FROMwordTOscale[Part[hw,i]],{i,1,Length[hw]}]

tonalitymembershipQ[hw_,t_]:=MemberQ[harmonicwords[t,Length[hw]],hw]
```

It may be useful to report some example:

Example VIII.3

TONALITIES OF THE MOST COMMON WORDS AT DIFFERENT LEVELS

```
tonality[majorword[0],1] =
```

```
{{0,4,7},{2,5,9},{4,7,11},{5,9,0},{7,11,2},{9,0,4},{11,2,5}}

tonality[majorword[0],2] =
{{0,4,7,11},{2,5,9,0},{4,7,11,2},{5,9,0,4},{7,11,2,5},{9,0,4,7},{11,2,5,9}}

tonality[majorword[0],3] =
{{0,4,7,11,2},{2,5,9,0,4},{4,7,11,2,5},{5,9,0,4,7},{7,11,2,5,9},{9,0,4,7,11},{
    11,2,5,9,0}}

tonality[majorword[0],4] =
{{0,4,7,11,2,5},{2,5,9,0,4,7},{4,7,11,2,5,9},{5,9,0,4,7,11},{7,11,2,5,9,0},{9,
    0,4,7,11,2},{11,2,5,9,0,4}}

tonality[majorword[0],5] =
{{0,4,7,11,2,5,9},{2,5,9,0,4,7,11},{4,7,11,2,5,9,0},{5,9,0,4,7,11,2},{7,11,2,
    5,9,0,4},{9,0,4,7,11,2,5},{11,2,5,9,0,4,7}}

tonality[minorword[0],1] =
{{0,3,7},{2,5,8},{3,7,10},{5,8,0},{7,10,2},{8,0,3},{10,2,5}}

tonality[minorword[0],2] =
{{0,3,7,10},{2,5,8,0},{3,7,10,2},{5,8,0,3},{7,10,2,5},{8,0,3,7},{10,2,5,8}}

tonality[minorword[0],3] =
{{0,3,7,10,2},{2,5,8,0,3},{3,7,10,2,5},{5,8,0,3,7},{7,10,2,5,8},{8,0,3,7,10},{
    10,2,5,8,0}}

tonality[minorword[0],4] =
{{0,3,7,10,2,5},{2,5,8,0,3,7},{3,7,10,2,5,8},{5,8,0,3,7,10},{7,10,2,5,8,0},{8,
    0,3,7,10,2},{10,2,5,8,0,3}}

tonality[minorword[0],5] =
```

```
{{0,3,7,10,2,5,8},{2,5,8,0,3,7,10},{3,7,10,2,5,8,0},{5,8,0,3,7,10,2},{7,10,2,
    5,8,0,3},{8,0,3,7,10,2,5},{10,2,5,8,0,3,7}}

tonality[harmonicminorword[0],1] =
{{0,3,7},{2,5,8},{3,7,11},{5,8,0},{7,11,2},{8,0,3},{11,2,5}}

tonality[harmonicminorword[0],2] =
{{0,3,7,11},{2,5,8,0},{3,7,11,2},{5,8,0,3},{7,11,2,5},{8,0,3,7},{11,2,5,8}}

tonality[harmonicminorword[0],3] =
{{0,3,7,11,2},{2,5,8,0,3},{3,7,11,2,5},{5,8,0,3,7},{7,11,2,5,8},{8,0,3,7,11},{
    11,2,5,8,0}}

tonality[harmonicminorword[0],4] =
{{0,3,7,11,2,5},{2,5,8,0,3,7},{3,7,11,2,5,8},{5,8,0,3,7,11},{7,11,2,5,8,0},{8,
    0,3,7,11,2},{11,2,5,8,0,3}}

tonality[harmonicminorword[0],5] =
{{0,3,7,11,2,5,8},{2,5,8,0,3,7,11},{3,7,11,2,5,8,0},{5,8,0,3,7,11,2},{7,11,2,
    5,8,0,3},{8,0,3,7,11,2,5},{11,2,5,8,0,3,7}}

tonality[dorianword[0],1] =
{{0,3,7},{2,5,9},{3,7,10},{5,9,0},{7,10,2},{9,0,3},{10,2,5}}

tonality[dorianword[0],2] =
{{0,3,7,10},{2,5,9,0},{3,7,10,2},{5,9,0,3},{7,10,2,5},{9,0,3,7},{10,2,5,9}}

tonality[dorianword[0],3] =
{{0,3,7,10,2},{2,5,9,0,3},{3,7,10,2,5},{5,9,0,3,7},{7,10,2,5,9},{9,0,3,7,10},{
    10,2,5,9,0}}

tonality[dorianword[0],4] =
```

```
{{0,3,7,10,2,5},{2,5,9,0,3,7},{3,7,10,2,5,9},{5,9,0,3,7,10},{7,10,2,5,9,0},{9,
    0,3,7,10,2},{10,2,5,9,0,3}}

tonality[dorianword[0],5] =
{{0,3,7,10,2,5,9},{2,5,9,0,3,7,10},{3,7,10,2,5,9,0},{5,9,0,3,7,10,2},{7,10,2,
    5,9,0,3},{9,0,3,7,10,2,5},{10,2,5,9,0,3,7}}

tonality[phrigianword[0],1] =
{{0,3,7},{1,5,8},{3,7,10},{5,8,0},{7,10,1},{8,0,3},{10,1,5}}

tonality[phrigianword[0],2] =
{{0,3,7,10},{1,5,8,0},{3,7,10,1},{5,8,0,3},{7,10,1,5},{8,0,3,7},{10,1,5,8}}

tonality[phrigianword[0],3] =
{{0,3,7,10,1},{1,5,8,0,3},{3,7,10,1,5},{5,8,0,3,7},{7,10,1,5,8},{8,0,3,7,10},{
    10,1,5,8,0}}

tonality[phrigianword[0],4] =
{{0,3,7,10,1,5},{1,5,8,0,3,7},{3,7,10,1,5,8},{5,8,0,3,7,10},{7,10,1,5,8,0},{8,
    0,3,7,10,1},{10,1,5,8,0,3}}

tonality[phrigianword[0],5] =
{{0,3,7,10,1,5,8},{1,5,8,0,3,7,10},{3,7,10,1,5,8,0},{5,8,0,3,7,10,1},{7,10,1,
    5,8,0,3},{8,0,3,7,10,1,5},{10,1,5,8,0,3,7}}

tonality[lydianword[0],1] =
{{0,4,7},{2,6,9},{4,7,11},{6,9,0},{7,11,2},{9,0,4},{11,2,6}}

tonality[lydianword[0],2] =
{{0,4,7,11},{2,6,9,0},{4,7,11,2},{6,9,0,4},{7,11,2,6},{9,0,4,7},{11,2,6,9}}

tonality[lydianword[0],3] =
```

```
{{0,4,7,11,2},{2,6,9,0,4},{4,7,11,2,6},{6,9,0,4,7},{7,11,2,6,9},{9,0,4,7,11},{
    11,2,6,9,0}}

tonality[lydianword[0],4] =
{{0,4,7,11,2,6},{2,6,9,0,4,7},{4,7,11,2,6,9},{6,9,0,4,7,11},{7,11,2,6,9,0},{9,
    0,4,7,11,2},{11,2,6,9,0,4}}

tonality[lydianword[0],5] =
{{0,4,7,11,2,6,9},{2,6,9,0,4,7,11},{4,7,11,2,6,9,0},{6,9,0,4,7,11,2},{7,11,2,
    6,9,0,4},{9,0,4,7,11,2,6},{11,2,6,9,0,4,7}}

tonality[mixolydianword[0],1] =
{{0,4,7},{2,5,9},{4,7,10},{5,9,0},{7,10,2},{9,0,4},{10,2,5}}

tonality[mixolydianword[0],2] =
{{0,4,7,10},{2,5,9,0},{4,7,10,2},{5,9,0,4},{7,10,2,5},{9,0,4,7},{10,2,5,9}}

tonality[mixolydianword[0],3] =
{{0,4,7,10,2},{2,5,9,0,4},{4,7,10,2,5},{5,9,0,4,7},{7,10,2,5,9},{9,0,4,7,10},{
    10,2,5,9,0}}

tonality[mixolydianword[0],4] =
{{0,4,7,10,2,5},{2,5,9,0,4,7},{4,7,10,2,5,9},{5,9,0,4,7,10},{7,10,2,5,9,0},{9,
    0,4,7,10,2},{10,2,5,9,0,4}}

tonality[mixolydianword[0],5] =
{{0,4,7,10,2,5,9},{2,5,9,0,4,7,10},{4,7,10,2,5,9,0},{5,9,0,4,7,10,2},{7,10,2,
    5,9,0,4},{9,0,4,7,10,2,5},{10,2,5,9,0,4,7}}

tonality[locrianword[0],1] =
{{0,3,6},{1,5,8},{3,6,10},{5,8,0},{6,10,1},{8,0,3},{10,1,5}}
```

```
tonality[locrianword[0],2] =
{{0,3,6,10},{1,5,8,0},{3,6,10,1},{5,8,0,3},{6,10,1,5},{8,0,3,6},{10,1,5,8}}

tonality[locrianword[0],3] =
{{0,3,6,10,1},{1,5,8,0,3},{3,6,10,1,5},{5,8,0,3,6},{6,10,1,5,8},{8,0,3,6,10},{
    10,1,5,8,0}}

tonality[locrianword[0],4] =
{{0,3,6,10,1,5},{1,5,8,0,3,6},{3,6,10,1,5,8},{5,8,0,3,6,10},{6,10,1,5,8,0},{8,
    0,3,6,10,1},{10,1,5,8,0,3}}

tonality[locrianword[0],5] =
{{0,3,6,10,1,5,8},{1,5,8,0,3,6,10},{3,6,10,1,5,8,0},{5,8,0,3,6,10,1},{6,10,1,
    5,8,0,3},{8,0,3,6,10,1,5},{10,1,5,8,0,3,6}}

tonality[tziganminorword[0],1] =
{{0,3,7},{2,6,8},{3,7,11},{6,8,0},{7,11,2},{8,0,3},{11,2,6}}

tonality[tziganminorword[0],2] =
{{0,3,7,11},{2,6,8,0},{3,7,11,2},{6,8,0,3},{7,11,2,6},{8,0,3,7},{11,2,6,8}}

tonality[tziganminorword[0],3] =
{{0,3,7,11,2},{2,6,8,0,3},{3,7,11,2,6},{6,8,0,3,7},{7,11,2,6,8},{8,0,3,7,11},{
    11,2,6,8,0}}

tonality[tziganminorword[0],4] =
{{0,3,7,11,2,6},{2,6,8,0,3,7},{3,7,11,2,6,8},{6,8,0,3,7,11},{7,11,2,6,8,0},{8,
    0,3,7,11,2},{11,2,6,8,0,3}}

tonality[tziganminorword[0],5] =
{{0,3,7,11,2,6,8},{2,6,8,0,3,7,11},{3,7,11,2,6,8,0},{6,8,0,3,7,11,2},{7,11,2,
    6,8,0,3},{8,0,3,7,11,2,6},{11,2,6,8,0,3,7}}
```

```
tonality[jewishword[0],1] =
{{0,4,7},{1,5,8},{4,7,10},{5,8,0},{7,10,1},{8,0,4},{10,1,5}}

tonality[jewishword[0],2] =
{{0,4,7,10},{1,5,8,0},{4,7,10,1},{5,8,0,4},{7,10,1,5},{8,0,4,7},{10,1,5,8}}

tonality[jewishword[0],3] =
{{0,4,7,10,1},{1,5,8,0,4},{4,7,10,1,5},{5,8,0,4,7},{7,10,1,5,8},{8,0,4,7,10},{
    10,1,5,8,0}}

tonality[jewishword[0],4] =
{{0,4,7,10,1,5},{1,5,8,0,4,7},{4,7,10,1,5,8},{5,8,0,4,7,10},{7,10,1,5,8,0},{8,
    0,4,7,10,1},{10,1,5,8,0,4}}

tonality[jewishword[0],5] =
{{0,4,7,10,1,5,8},{1,5,8,0,4,7,10},{4,7,10,1,5,8,0},{5,8,0,4,7,10,1},{7,10,1,
    5,8,0,4},{8,0,4,7,10,1,5},{10,1,5,8,0,4,7}}

tonality[majorpentatonicword[0],1] =
{{0,4,9},{2,7,0},{4,9,2},{7,0,4},{9,2,7}}

tonality[majorpentatonicword[0],2] =
{{0,4,9,2},{2,7,0,4},{4,9,2,7},{7,0,4,9},{9,2,7,0}}

tonality[majorpentatonicword[0],3] =
{{0,4,9,2,7},{2,7,0,4,9},{4,9,2,7,0},{7,0,4,9,2},{9,2,7,0,4}}

tonality[minorpentatonicword[0],1] =
{{3,7,0},{5,10,3},{7,0,5},{10,3,7},{0,5,10}}

tonality[minorpentatonicword[0],2] =
```

```
{{3,7,0,5},{5,10,3,7},{7,0,5,10},{10,3,7,0},{0,5,10,3}}

tonality[minorpentatonicword[0],3] =
{{3,7,0,5,10},{5,10,3,7,0},{7,0,5,10,3},{10,3,7,0,5},{0,5,10,3,7}}

tonality[bluesword[0],1] =
{{0,5,7},{3,6,10},{5,7,0},{6,10,3},{7,0,5},{10,3,6}}

tonality[esatonalword[0],1] =
{{0,4,8},{2,6,10},{4,8,0},{6,10,2},{8,0,4},{10,2,6}}

tonality[augmentedword[0],1] =
{{0,4,8},{3,7,11},{4,8,0},{7,11,3},{8,0,4},{11,3,7}}

tonality[halfwholediminishedword[0],1] =
{{0,3,6},{1,4,7},{3,6,9},{4,7,10},{6,9,0},{7,10,1},{9,0,3},{10,1,4}}

tonality[halfwholediminishedword[0],2] =
{{0,3,6,9},{1,4,7,10},{3,6,9,0},{4,7,10,1},{6,9,0,3},{7,10,1,4},{9,0,3,6},{10,
    1,4,7}}

tonality[bebopmajorword[0],1] =
{{0,4,7},{2,5,8},{4,7,9},{5,8,11},{7,9,0},{8,11,2},{9,0,4},{11,2,5}}

tonality[bebopmajorword[0],2] =
{{0,4,7,9},{2,5,8,11},{4,7,9,0},{5,8,11,2},{7,9,0,4},{8,11,2,5},{9,0,4,7},{11,
    2,5,8}}

tonality[bebopdominant[0],1] =
{{0,4,7},{2,5,9},{4,7,10},{5,9,11},{7,10,0},{9,11,2},{10,0,4},{11,2,5}}

tonality[bebopdominant[0],2] =
```

```
{{0,4,7,10},{2,5,9,11},{4,7,10,0},{5,9,11,2},{7,10,0,4},{9,11,2,5},{10,0,4,
    7},{11,2,5,9}}

tonality[chromaticword[0],1] =
{{0,2,4},{1,3,5},{2,4,6},{3,5,7},{4,6,8},{5,7,9},{6,8,10},{7,9,11},{8,10,0},{
    9,11,1},{10,0,2},{11,1,3}}

tonality[chromaticword[0],2] =
{{0,2,4,6},{1,3,5,7},{2,4,6,8},{3,5,7,9},{4,6,8,10},{5,7,9,11},{6,8,10,0},{7,
    9,11,1},{8,10,0,2},{9,11,1,3},{10,0,2,4},{11,1,3,5}}

tonality[chromaticword[0],3] =
{{0,2,4,6,8},{1,3,5,7,9},{2,4,6,8,10},{3,5,7,9,11},{4,6,8,10,0},{5,7,9,11,1},{

6,8,10,0,2},{7,9,11,1,3},{8,10,0,2,4},{9,11,1,3,5},{10,0,2,4,6},{11,1,3,5,
    7}}

tonality[chromaticword[0],4] =
{{0,2,4,6,8,10},{1,3,5,7,9,11},{2,4,6,8,10,0},{3,5,7,9,11,1},{4,6,8,10,0,2},{
    5,7,9,11,1,3},{6,8,10,0,2,4},{7,9,11,1,3,5},{8,10,0,2,4,6},{9,11,1,3,5,
    7},{10,0,2,4,6,8},{11,1,3,5,7,9}}
```

Example VIII.4

THE II-V-I PROGRESSION

Let us consider the II-V-I progression, the most usual progression in Jazz [77]:

```
harmonicwordintermofdegrees[tonality[majorword[0],1],{2,5,1}] =
{{2,5,9},{7,11,2},{0,4,7}}

harmonicwordintermofdegrees[tonality[majorword[0],2],{2,5,1}] =
```

{{2,5,9,0},{7,11,2,5},{0,4,7,11}}

harmonicwordintermofdegrees[tonality[majorword[0],3],{2,5,1}] =
{{2,5,9,0,4},{7,11,2,5,9},{0,4,7,11,2}}

harmonicwordintermofdegrees[tonality[majorword[0],4],{2,5,1}] =
{{2,5,9,0,4,7},{7,11,2,5,9,0},{0,4,7,11,2,5}}

harmonicwordintermofdegrees[tonality[majorword[0],5],{2,5,1}] =
{{2,5,9,0,4,7,11},{7,11,2,5,9,0,4},{0,4,7,11,2,5,9}}

harmonicwordintermofdegrees[tonality[minorword[0],1],{2,5,1}] =
{{2,5,8},{7,10,2},{0,3,7}}

harmonicwordintermofdegrees[tonality[minorword[0],2],{2,5,1}] =
{{2,5,8,0},{7,10,2,5},{0,3,7,10}}

harmonicwordintermofdegrees[tonality[minorword[0],3],{2,5,1}] =
{{2,5,8,0,3},{7,10,2,5,8},{0,3,7,10,2}}

harmonicwordintermofdegrees[tonality[minorword[0],4],{2,5,1}] =
{{2,5,8,0,3,7},{7,10,2,5,8,0},{0,3,7,10,2,5}}

harmonicwordintermofdegrees[tonality[minorword[0],5],{2,5,1}] =
{{2,5,8,0,3,7,10},{7,10,2,5,8,0,3},{0,3,7,10,2,5,8}}

harmonicwordintermofdegrees[tonality[harmonicminorword[0],1],{2,5,1}]
= {{2,5,8},{7,11,2},{0,3,7}}

harmonicwordintermofdegrees[tonality[harmonicminorword[0],2],{2,5,1}]
= {{2,5,8,0},{7,11,2,5},{0,3,7,11}}

```
harmonicwordintermofdegrees[tonality[harmonicminorword[0],3],{2,5,1}]
= {{2,5,8,0,3},{7,11,2,5,8},{0,3,7,11,2}}

harmonicwordintermofdegrees[tonality[harmonicminorword[0],4],{2,5,1}]
= {{2,5,8,0,3,7},{7,11,2,5,8,0},{0,3,7,11,2,5}}

harmonicwordintermofdegrees[tonality[harmonicminorword[0],5],{2,5,1}]
= {{2,5,8,0,3,7,11},{7,11,2,5,8,0,3},{0,3,7,11,2,5,8}}

harmonicwordintermofdegrees[tonality[dorianword[0],1],{2,5,1}] =
{{2,5,9},{7,10,2},{0,3,7}}

harmonicwordintermofdegrees[tonality[dorianword[0],2],{2,5,1}] =
{{2,5,9,0},{7,10,2,5},{0,3,7,10}}

harmonicwordintermofdegrees[tonality[dorianword[0],3],{2,5,1}] =
{{2,5,9,0,3},{7,10,2,5,9},{0,3,7,10,2}}

harmonicwordintermofdegrees[tonality[dorianword[0],4],{2,5,1}] =
{{2,5,9,0,3,7},{7,10,2,5,9,0},{0,3,7,10,2,5}}

harmonicwordintermofdegrees[tonality[dorianword[0],5],{2,5,1}] =
{{2,5,9,0,3,7,10},{7,10,2,5,9,0,3},{0,3,7,10,2,5,9}}

harmonicwordintermofdegrees[tonality[phrigianword[0],1],{2,5,1}] =
{{1,5,8},{7,10,1},{0,3,7}}

harmonicwordintermofdegrees[tonality[phrigianword[0],2],{2,5,1}] =
{{1,5,8,0},{7,10,1,5},{0,3,7,10}}

harmonicwordintermofdegrees[tonality[phrigianword[0],3],{2,5,1}] =
{{1,5,8,0,3},{7,10,1,5,8},{0,3,7,10,1}}
```

```
harmonicwordintermofdegrees[tonality[phrigianword[0],4],{2,5,1}] =
{{1,5,8,0,3,7},{7,10,1,5,8,0},{0,3,7,10,1,5}}

harmonicwordintermofdegrees[tonality[phrigianword[0],5],{2,5,1}] =
{{1,5,8,0,3,7,10},{7,10,1,5,8,0,3},{0,3,7,10,1,5,8}}

harmonicwordintermofdegrees[tonality[lydianword[0],1],{2,5,1}] =
{{2,6,9},{7,11,2},{0,4,7}}

harmonicwordintermofdegrees[tonality[lydianword[0],2],{2,5,1}] =
{{2,6,9,0},{7,11,2,6},{0,4,7,11}}

harmonicwordintermofdegrees[tonality[lydianword[0],3],{2,5,1}] =
{{2,6,9,0,4},{7,11,2,6,9},{0,4,7,11,2}}

harmonicwordintermofdegrees[tonality[lydianword[0],4],{2,5,1}] =
{{2,6,9,0,4,7},{7,11,2,6,9,0},{0,4,7,11,2,6}}

harmonicwordintermofdegrees[tonality[lydianword[0],5],{2,5,1}] =
{{2,6,9,0,4,7,11},{7,11,2,6,9,0,4},{0,4,7,11,2,6,9}}

harmonicwordintermofdegrees[tonality[mixolydianword[0],1],{2,5,1}]
= {{2,5,9},{7,10,2},{0,4,7}}

harmonicwordintermofdegrees[tonality[mixolydianword[0],2],{2,5,1}]
= {{2,5,9,0},{7,10,2,5},{0,4,7,10}}

harmonicwordintermofdegrees[tonality[mixolydianword[0],3],{2,5,1}]
= {{2,5,9,0,4},{7,10,2,5,9},{0,4,7,10,2}}

harmonicwordintermofdegrees[tonality[mixolydianword[0],4],{2,5,1}]
```

```
= {{2,5,9,0,4,7},{7,10,2,5,9,0},{0,4,7,10,2,5}}

harmonicwordintermofdegrees[tonality[mixolydianword[0],5],{2,5,1}]
= {{2,5,9,0,4,7,10},{7,10,2,5,9,0,4},{0,4,7,10,2,5,9}}

harmonicwordintermofdegrees[tonality[locrianword[0],1],{2,5,1}] =
{{1,5,8},{6,10,1},{0,3,6}}

harmonicwordintermofdegrees[tonality[locrianword[0],2],{2,5,1}] =
{{1,5,8,0},{6,10,1,5},{0,3,6,10}}

harmonicwordintermofdegrees[tonality[locrianword[0],3],{2,5,1}] =
{{1,5,8,0,3},{6,10,1,5,8},{0,3,6,10,1}}

harmonicwordintermofdegrees[tonality[locrianword[0],4],{2,5,1}] =
{{1,5,8,0,3,6},{6,10,1,5,8,0},{0,3,6,10,1,5}}

harmonicwordintermofdegrees[tonality[locrianword[0],5],{2,5,1}] =
{{1,5,8,0,3,6,10},{6,10,1,5,8,0,3},{0,3,6,10,1,5,8}}

harmonicwordintermofdegrees[tonality[tziganminorword[0],1],{2,5,1}]
= {{2,6,8},{7,11,2},{0,3,7}}

harmonicwordintermofdegrees[tonality[tziganminorword[0],2],{2,5,1}]
= {{2,6,8,0},{7,11,2,6},{0,3,7,11}}

harmonicwordintermofdegrees[tonality[tziganminorword[0],3],{2,5,1}]
= {{2,6,8,0,3},{7,11,2,6,8},{0,3,7,11,2}}

harmonicwordintermofdegrees[tonality[tziganminorword[0],4],{2,5,1}]
= {{2,6,8,0,3,7},{7,11,2,6,8,0},{0,3,7,11,2,6}}
```

```
harmonicwordintermofdegrees[tonality[tziganminorword[0],5],{2,5,1}]
= {{2,6,8,0,3,7,11},{7,11,2,6,8,0,3},{0,3,7,11,2,6,8}}

harmonicwordintermofdegrees[tonality[jewishword[0],1],{2,5,1}] =
{{1,5,8},{7,10,1},{0,4,7}}

harmonicwordintermofdegrees[tonality[jewishword[0],2],{2,5,1}] =
{{1,5,8,0},{7,10,1,5},{0,4,7,10}}

harmonicwordintermofdegrees[tonality[jewishword[0],3],{2,5,1}] =
{{1,5,8,0,4},{7,10,1,5,8},{0,4,7,10,1}}

harmonicwordintermofdegrees[tonality[jewishword[0],4],{2,5,1}] =
{{1,5,8,0,4,7},{7,10,1,5,8,0},{0,4,7,10,1,5}}

harmonicwordintermofdegrees[tonality[jewishword[0],5],{2,5,1}] =
{{1,5,8,0,4,7,10},{7,10,1,5,8,0,4},{0,4,7,10,1,5,8}}

harmonicwordintermofdegrees[tonality[majorpentatonicword[0],1],{2,5,1}]
= {{2,7,0},{9,2,7},{0,4,9}}

harmonicwordintermofdegrees[tonality[majorpentatonicword[0],2],{2,5,1}]
= {{2,7,0,4},{9,2,7,0},{0,4,9,2}}

harmonicwordintermofdegrees[tonality[majorpentatonicword[0],3],{2,5,1}]
= {{2,7,0,4,9},{9,2,7,0,4},{0,4,9,2,7}}

harmonicwordintermofdegrees[tonality[minorpentatonicword[0],1],{2,5,1}]
= {{5,10,3},{0,5,10},{3,7,0}}

harmonicwordintermofdegrees[tonality[minorpentatonicword[0],2],{2,5,1}]
= {{5,10,3,7},{0,5,10,3},{3,7,0,5}}
```

```
harmonicwordintermofdegrees[tonality[minorpentatonicword[0],3],{2,5,1}]
= {{5,10,3,7,0},{0,5,10,3,7},{3,7,0,5,10}}

harmonicwordintermofdegrees[tonality[bluesword[0],1],{2,5,1}] =
{{3,6,10},{7,0,5},{0,5,7}}

harmonicwordintermofdegrees[tonality[esatonalword[0],1],{2,5,1}] =
{{2,6,10},{8,0,4},{0,4,8}}

harmonicwordintermofdegrees[tonality[augmentedword[0],1],{2,5,1}]
= {{3,7,11},{8,0,4},{0,4,8}}

harmonicwordintermofdegrees[tonality[halfwholediminishedword[0],1],{2,5,1}]
= {{1,4,7},{6,9,0},{0,3,6}}

harmonicwordintermofdegrees[tonality[halfwholediminishedword[0],2],{2,5,1}]
= {{1,4,7,10},{6,9,0,3},{0,3,6,9}}

harmonicwordintermofdegrees[tonality[wholehalfdiminishedword[0],1],{2,5,1}]
= {{2,5,8},{6,9,0},{0,3,6}}

harmonicwordintermofdegrees[tonality[wholehalfdiminishedword[0],2],{2,5,1}]
= {{2,5,8,11},{6,9,0,3},{0,3,6,9}}

harmonicwordintermofdegrees[tonality[wholetonediminishedword[0],1],{2,5,1}]
= {{1,4,8},{6,10,1},{0,3,6}}

harmonicwordintermofdegrees[tonality[wholetonediminishedword[0],2],{2,5,1}]
= {{1,4,8,0},{6,10,1,4},{0,3,6,10}}

harmonicwordintermofdegrees[tonality[wholetonediminishedword[0],3],{2,5,1}]
```

```
= {{1,4,8,0,3},{6,10,1,4,8},{0,3,6,10,1}}

harmonicwordintermofdegrees[tonality[wholetonediminishedword[0],4],{2,5,1}]
= {{1,4,8,0,3,6},{6,10,1,4,8,0},{0,3,6,10,1,4}}

harmonicwordintermofdegrees[tonality[wholetonediminishedword[0],5],{2,5,1}]
= {{1,4,8,0,3,6,10},{6,10,1,4,8,0,3},{0,3,6,10,1,4,8}}

harmonicwordintermofdegrees[tonality[bebopmajorword[0],1],{2,5,1}]
= {{2,5,8},{7,9,0},{0,4,7}}

harmonicwordintermofdegrees[tonality[bebopmajorword[0],2],{2,5,1}]
= {{2,5,8,11},{7,9,0,4},{0,4,7,9}}

harmonicwordintermofdegrees[tonality[bebopdominant[0],1],{2,5,1}]
= {{2,5,9},{7,10,0},{0,4,7}}

harmonicwordintermofdegrees[tonality[bebopdominant[0],2],{2,5,1}]
= {{2,5,9,11},{7,10,0,4},{0,4,7,10}}

harmonicwordintermofdegrees[tonality[chromaticword[0],1],{2,5,1}]
= {{1,3,5},{4,6,8},{0,2,4}}

harmonicwordintermofdegrees[tonality[chromaticword[0],2],{2,5,1}]
= {{1,3,5,7},{4,6,8,10},{0,2,4,6}}

harmonicwordintermofdegrees[tonality[chromaticword[0],3],{2,5,1}]
= {{1,3,5,7,9},{4,6,8,10,0},{0,2,4,6,8}}

harmonicwordintermofdegrees[tonality[chromaticword[0],4],{2,5,1}]
= {{1,3,5,7,9,11},{4,6,8,10,0,2},{0,2,4,6,8,10}}
```

Given two tonalities t_1 , $t_2 \in \mathcal{T}$:

DEFINITION VIII.37

PIVOTAL DEGREES OF t_1 AND t_2:

$$\mathcal{P}(t_1 \, , \, t_2) \; := \; Range(chord^{(level(t_1)}(t_1)) \bigcap Range(chord^{(level(t_2))}(t_2)) \tag{8.81}$$

Such a concept, essential as to Modulation Theory, may be easily computed through the following Mathematica expression from the notebook of sectionA:

```
pivotaldegrees[t1_,t2_]:=Intersection[t1,t2]

(*** "pivotaldegreesintermofdegrees[t1,t2]" give a list of two
elements,  the first being the pivotal degrees of "t1" and "t2"
expressed as degrees of "t1", the second being the pivotal degrees
of "t1" and "t2" expressed as degrees of t2" ***)

pivotaldegreesintermofdegrees[t1_,t2_]:={
    Table[degreeofachordinatonality[Part[pivotaldegrees[t1,t2],i],t1],{i,1,
        Length[pivotaldegrees[t1,t2]]}],
    Table[degreeofachordinatonality[Part[pivotaldegrees[t1,t2],i],t2],{i,1,
        Length[pivotaldegrees[t1,t2]]}]}
```

It is important to remark that, obviously, the pivotal degrees depends crucially from the level at which the involved tonalities are built:

Example VIII.5

PIVOTAL DEGREES OF MAJOR TONALITIES AT DIFFERENT LEVELS ALONG A CLOCKWISE STEP IN THE FIFTHS' CYCLE

$$pivotaldegreesintermofdegrees[tonality[majorword[0],1], tonality[majorword[7],1]] = \{\{1,3,5,6\},\{4,6,1,2\}\} \tag{8.82}$$

$$pivotaldegreesintermofdegrees[tonality[majorword[0],2],tonality[majorword[7],2]] = \{\{1,3,6\},\{4,6,2\}\} \quad (8.83)$$

$$pivotaldegreesintermofdegrees[tonality[majorword[0],3],tonality[majorword[7],3]] = \{\{1,6\},\{4,2\}\} \quad (8.84)$$

$$pivotaldegreesintermofdegrees[tonality[majorword[0],4],tonality[majorword[7],4]] = \{\{6\},\{2\}\} \quad (8.85)$$

$$pivotaldegreesintermofdegrees[tonality[majorword[0],5],tonality[majorword[7],5]] = \emptyset \quad (8.86)$$

so that, as to Classical Harmony, the $I_C = IV_G = C$, the $III_C = VI_G = Em$, the $V_C = I_G = G7$, the $VI_C = II_G = Am$ are four different streets one can follow to modulate from C Major to G Major, a thing almost forbidden in that context.

In Jazz Harmony, where parallel motion is allowed, tonalities are defined at a level > 1 with the number of possible streets lowering with the considered level: at second level one still has that [19] $I^7_C = IV^7_G = C7+$, $III^7_C = VI^7_G = Em7$, at third level remains only that $I^9_C = IV^9_G = C7+/9$ and that $VI^9_C = II^9_G = Am9$, at fourth level remains only that $VI^{11}_C = II^{11}_G = Am11$.

Example VIII.6

PIVOTAL DEGREES OF JEWISH TONALITIES AT DIFFERENT LEVELS ALONG A CLOCKWISE STEP IN THE FIFTHS' CYCLE

```
pivotaldegreesintermofdegrees[tonality[jewishword[0],1],
  tonality[jewishword[7],1]] = {{4},{7}}


pivotaldegreesintermofdegrees[tonality[jewishword[0],2],
  tonality[jewishword[7],2]] = {{},{}}
```

[19] As to the nomenclature for chords I will adopt that of American Jazz Theory exposed in the appendix L.2 "Third Chain Classes" of [9]

so that parallel motion of a clockwise step in the fifths' cycle has strongly less chances for jewish tonalities than for the major tonalities discussed in the example VIII.5: at the first level one has the only pivotal degree $IV_C = VII_G = Fm$ while no pivotal degree exists already at the second level.

Let us now pass to analyze the concept of cadence: with an eye to its application as to Modulation Theory, Mazzola (cfr. the 26^{th} chapter "Cadences" of [9]) has correctly catched the structural peculiarity of such harmonic words: they identify a tonality among a suitable class of tonalities.

What is important, at this point, to stress is how such a concept critically depends by a context of tonalities among which the cadential one has to identify its tonality.

Mazzola's definition of cadence may be easily translated also as to our more general definition of tonality, namely the definitionVIII.34, in the following way:

given a tonality $t \in \mathcal{T}$ and a context $\mathcal{T}_{context} \subseteq \mathcal{T}$

DEFINITION VIII.38

CADENCES OF t W.R.T. THE CONTEXT $\mathcal{T}_{context}$

$$\mathcal{C}(t\,,\,\mathcal{T}_{context}) \;:=\; \{c \,\in\, \mathcal{HW}(t) \;:\; (\,c \,\in\, \mathcal{HW}(t) \;\Rightarrow\; t \,=\, u\,)\; \forall u \in \mathcal{T}_{context}\} \tag{8.87}$$

To understand how crucial is the role of the context in the definitionVIII.38 let us observe that:

Theorem VIII.4

NO-GO THEOREM FOR CADENCES IN A MULTI-MODE CONTEXT

HP:

$$\vec{x}_1\,,\,\vec{x}_2 \,\in\, \mathbb{Z}_{12}^{\star} \tag{8.88}$$

$$\mathcal{T}_{context} \subset \mathcal{T} \;:\; tonality(\vec{x}_1, n), tonality(\vec{x}_2, n) \in \mathcal{T}_{context} \tag{8.89}$$

$$\exists i \;: \vec{x}_2 = mode(\vec{x}_1, i) \tag{8.90}$$

TH:

$$\mathcal{C}[tonality(\vec{x}_1 \, , \, n) \, , \, \mathcal{T}_{context}] \; = \; \emptyset \tag{8.91}$$

PROOF:

Since:

$$chord(\vec{x}_2, j, n) \; = \; chord(\vec{x}_2, Mod[j+i, |\vec{x_1}|], n) \;\; \forall j = 1, \cdots, |\vec{x_2}| \tag{8.92}$$

one has that:

$$hw \in tonality[\vec{x_1}, n] \; \Rightarrow \; hw \in tonality[\vec{x_2}, n] \; \Rightarrow \; hw \notin \mathcal{C}(tonality[\vec{x}_1 \, , \, n] \, , \, \mathcal{T}_{context}) \tag{8.93}$$

■

An immediate consequence of the theorem VIII.4 is the following:

Corollary VIII.3

$$\mathcal{C}(t \, , \, \mathcal{T}_{greg}) \; = \; \emptyset \;\; \forall t \in \mathcal{T}_{greg} \tag{8.94}$$

$$\mathcal{C}(t \, , \, \mathcal{T}_{cl}) \; = \; \emptyset \;\; \forall t \in \mathcal{T}_{cl} \tag{8.95}$$

A someway natural choice of the context respect to which to analyze whether an harmonic word is a tonality are translational equivalence classes, a fact this one we will analyze more deeply when we will discuss Modulation Theory.

Such an idea may be formalized, anyway, in the following way:

given a word $\vec{x} \in \Sigma^{\star}$ a natural number $n \in \mathbb{N} \, : \, n < maxlevel(\vec{x})$

DEFINITION VIII.39

NATURAL CONTEXT OF $tonality[\vec{x}, n]$

$$\mathcal{T}_{n.c.}(\vec{x}, n) \; := \; \{tonality(\vec{y} \, , \, n) \; : \; \vec{x} \; \sim_T \; \vec{y}\} \tag{8.96}$$

Given a tonality $t \in \mathcal{T}$ and a context $\mathcal{T}_{context} \subseteq \mathcal{T}$

DEFINITION VIII.40

MINIMAL CADENCES OF t W.R.T. THE CONTEXT $\mathcal{T}_{context}$

$$\mathcal{MC}(t \, , \, \mathcal{T}_{context}) \; := \; \{c_1 \; \in \; \mathcal{C}(t, \mathcal{T}_{context}) \; : \; \nexists c_2 \in \mathcal{C}(t, \mathcal{T}_{context}), c_2 \; <_p \; c_1\} \tag{8.97}$$

Cadences and minimal cadences with respect to suitable contexts may be easily computed through the following expressions from the Mathematica notebook of section A:

```
setofthemajortonalities[n_]:=Table[tonality[majorword[i],n],{i,0,11}];

setoftheminortonalities[n_]:=Table[tonality[minorword[i],n],{i,0,11}];

setoftheclassicaltonalities[n_]:=
  Union[setofthemajortonalities[n],setoftheminortonalities[n]];

setofthegregoriantonalities[n_]:=
  Flatten[Table[tonality[mode[majorword[i],j],n],{j,1,7},{i,0,11}],1];

setofthemazzolatonalities[n_]:=

Table[tonality[Part[mazzolawords,i],n],{i,1,Length[nonrepetitivewords[7]]}];

setofthejewishtonalities[n_]:=Table[tonality[jewishword[i],n],{i,0,11}];

truthQ[x_]:=Equal[x,True]

cadenceQ[hw_,setoftonalities_]:= If[Length[ Select[Table[
            tonalitymembershipQ[hw,Part[setoftonalities,i]],{i,1,
              Length[setoftonalities]}],truthQ]]==1,True,False]

subharmonicwords[hw_]:=LexicographicSubsets[hw]

cadences[t_,n_,setoftonalities_]:=
  generalizedselect[harmonicwordsupto[t,n],cadenceQ,setoftonalities]

cadencesintermofdegrees[t_,n_,setoftonalities_]:=
  Table[degreeofachordinatonality[
      Part[Part[cadences[t,n,setoftonalities],i],j],t],{i,1,
      Length[cadences[t,n,setoftonalities] ]},{j,1,
```

```
      Length[ Part[cadences[t,n,setoftonalities],i] ]}]

minimalcadenceQ[hw_,setoftonalities_]:=And[cadenceQ[hw,setoftonalities],

Equal[Table[cadenceQ[prefix[hw,i],setoftonalities],{i,1,Length[hw]-1}],
      Table[False,{Length[hw]-1}]]]

minimalcadences[t_,n_,setoftonalities_]:=
  generalizedselect[harmonicwords[t,n],minimalcadenceQ,setoftonalities]

minimalcadencesintermofdegrees[t_,n_,setoftonalities_]:=
  Table[degreeofachordinatonality[
      Part[Part[minimalcadences[t,n,setoftonalities],i],j],t],{i,1,
      Length[minimalcadences[t,n,setoftonalities] ]},{j,1,
      Length[ Part[minimalcadences[t,n,setoftonalities],i] ]}]
```

Remark VIII.2

THE PERFECT AND PLAGAL HARMONIC PROGRESSIONS:

Though expressing the idea that the role of a cadence is, in certain contexts, that of declaring the underlying tonality, hystorical tradition has led most of the more common Harmony's manuals [81], [82], [26] to speak about the so-called perfect cadence $V - I$ and plagal cadence $IV - I$. That such harmonic words are not cadences even with respect to the little class of major tonalities may be easily verified observing that, for example, the harmonic word $(V_C \, , \, I_C) \; = \; 7, 11, 2, 0, 4, 7 \; = \; (G \, , \, C)$ may be seen also as $(I_G \, , \, IV_G)$ and that the harmonic word $(IV_C \, , \, I_C) \, = \, 5, 9, 0, 0, 4, 7 \, = \, (F \, , \, C)$ may be seen also as $(I_F \, , \, V_F)$. Such a situation, anyway, changes if one takes in consideration major tonalities at a livel greater than the first.

Example VIII.7

CADENCES AND MINIMAL CADENCES OF THE C MAJOR TONALITY W.R.T. THE CONTEXT OF MAJOR TONALITIES AT DIFFERENT LEVELS

Obviously:

```
cadencesintermofdegrees[tonality[majorword[0],1],1,setoftheclassicaltonalities[1]]
=  {}
```

since any one-letter harmonic word in a major tonality x may be also seen as a one letter harmonic word in the relative minor tonality.

Restricting the set of the tonalities to the major ones one finds that:

```
cadencesintermofdegrees[tonality[majorword[0],1],1,setofthemajortonalities[1]]
= {{7}}
```

showing that VII_C is the only one-letter harmonic word to be a cadence w.r.t. the set of major tonalities at first level since it is the only diminished triad ($\{11, 2, 5\} = Bm5-$) in tonality[majorword[0],1], the other degrees being major triads ($\{0, 4, 7\} = C$, $\{5, 9, 0\} = F$, $\{7, 11, 2\} = G$) or minor triads ($\{2, 5, 9\} = Dm$, $\{4, 7, 11\} = Em$, $\{9, 0, 4\} = Am$).

Raising the level of the tonalities, anyway, the situation changes:

```
cadencesintermofdegrees[tonality[majorword[0],2],1,setofthemajortonalities[2]]
{{5},{7}}
```

since at the second level $V_C^{(7)} = \{7, 11, 2, 5\} = G7 =$ is the only first kind major seventh chord, $I_C^{(7)} = \{0, 4, 7, 11\} = C7+$ and $IV_C^{(7)} = \{5, 9, 0, 4\} = F7+$ being fourth kind seventh chords.

Passing to the third level:

```
cadencesintermofdegrees[tonality[majorword[0],3],1,setofthemajortonalities[3]]
= {{3},{5},{7}}
```

one obtains a new one-letter cadential set, i.e. $III_C^{(9)} = \{4, 7, 11, 2, 5\} = Em9-$, that is a ninth chord of different kind w.r.t. $II_C^{(9)} = \{2, 5, 9, 0, 4\} = Dm9$ and $VI_C^{(9)} = \{9, 0, 4, 7, 11\} = Am9$.

Passing to the fourth level:

```
cadencesintermofdegrees[tonality[majorword[0],4],1,setofthemajortonalities[4]]
= {{1},{3},{4},{5},{7}}
```

one obtains two new one-letter cadences, i.e. $I_C^{(11)} = \{0, 4, 7, 11, 2, 5\} = C7 + /11$ and $IV_C^{(11)} = \{5, 9, 0, 4, 7, 11\} = C7 + /11+$ that are 11^{th} chords of different type.

Passing, finally, to the fifth level:

```
cadencesintermofdegrees[tonality[majorword[0],5],1,setofthemajortonalities[5]]
= {{1},{2},{3},{4},{5},{6},{7}}
```

one can that all the one-letter harmonic words become cadential.

Let us now pass to analyze harmonic words of length two, restricting the analysis to the minimal cadences.

At the 1^{th} level:

```
minimalcadencesintermofdegrees[tonality[majorword[0],1],2,
  setofthemajortonalities[1]] =
{{1,7},{2,3},{2,5},{2,7},{3,2},{3,4},{3,7},{4,3},{4,5},{4,7},{5,2},{5,4},{5,
    7},{6,7}}
```

giving us confirmation of what we saw in the remarkVIII.2.

At the 2^{th} level:

```
minimalcadencesintermofdegrees[tonality[majorword[0],2],2,
  setofthemajortonalities[2]] =
{{1,2},{1,4},{1,5},{1,7},{2,1},{2,3},{2,5},{2,7},{3,2},{3,4},{3,5},{3,7},{4,
    1},{4,3},{4,5},{4,7},{6,5},{6,7}}
```

one sees that the plagal harmonic word $(IV_C^{(7)}, I_C^7) = \{\{5,9,0,4\},\{0,4,7,11\}\} = (F7+, C7+)$ becomes a cadence.

At the 3^{th} level:

```
minimalcadencesintermofdegrees[tonality[majorword[0],3],2,
  setofthemajortonalities[3]] =
{{1,2},{1,3},{1,4},{1,5},{1,7},{2,1},{2,3},{2,5},{2,6},{2,7},{4,1},{4,3},{4,
    5},{4,6},{4,7},{6,2},{6,3},{6,4},{6,5},{6,7}}
```

this is still true, while at the 4^{th} level:

```
minimalcadencesintermofdegrees[tonality[majorword[0],4],2,
  setofthemajortonalities[4]] =
{{2,1},{2,3},{2,4},{2,5},{2,6},{2,7},{6,1},{6,2},{6,3},{6,4},{6,5},{6,7}}
```

this is no longer true, since, though remaining a cadence:

```
cadenceQ[harmonicwordintermofdegrees[tonality[majorword[0],4],{4,1}],
  setofthemajortonalities[4]] = True
```

the plagal cadence is no more minimal since we saw previously that, at this level VI_C^7 is a cadence by itself. For the same reason:

```
minimalcadencesintermofdegrees[tonality[majorword[0],5],2,
  setofthemajortonalities[5]] = {}
```

no cadential set at the 5^{th} level can be minimal.

Example VIII.8

CADENCES AND MINIMAL CADENCES OF THE C JEWISH TONALITY W.R.T. THE CONTEXT OF JEWISH TONALITIES AT DIFFERENT LEVELS

At the 1^{th} level one has that:

```
cadencesintermofdegrees[tonality[jewishword[0],1],1,setofthejewishtonalities[1]]
= {{6}}
```

so that the only one-letter cadential set of the C Jewish tonality is $VI_C = \{8,0,4\} = G^{\sharp}5+$.

It is sufficient, anyway, to pass to the 2^{th} level:

```
cadencesintermofdegrees[tonality[jewishword[0],2],1,
  setofthejewishtonalities[2]] = {{1},{2},{3},{4},{5},{6},{7}}
```

that any one-letter harmonic-word becomes a cadence, such a property continuing, clearly, to hold at higher levels.

Passing to harmonic words of two letters one has that:

```
minimalcadencesintermofdegrees[tonality[jewishword[0],1],2,
  setofthejewishtonalities[1]] =
{{1,2},{1,3},{1,4},{1,5},{1,6},{1,7},{2,1},{2,3},{2,4},{2,5},{2,6},{2,7},{3,1},
{3,2},{3,4},{3,5},{3,6},{3,7},{4,1},{4,2},{4,3},{4,5},{4,6},{4,7},{5,1},
{5,2},{5,3},{5,4},{5,6},{5,7},{7,1},{7,2},{7,3},{7,4},{7,5},{7,6}}
```

It is interesting to note that, in particular, that both the perfect harmonic word $(V_C, I_C) = (\{7, 10, 1\}, \{0, 4, 7\}) = (Gm5-, C)$ and the plagal harmonic word $(IV_C, I_C) = (\{5, 8, 0\}, \{0, 4, 7\} = (Fm, C)$ are cadential.

Let us now analyze the Muzzulini - Mazzola's theory of modulations in the framework of Symmetry's Theory [83], [9].

Given an arbitrary finite alphabet Σ, a word $\vec{x} \in \Sigma^{\star}$ and a map $g : \Sigma \rightarrow \Sigma$:

DEFINITION VIII.41

g IS A SYMMETRY OF $\vec{x}$:

$$\hat{g}(\vec{x}) = \vec{x} \tag{8.98}$$

The symmetries of a word $\vec{x}$ constitute a group I will denote as $SYM(\vec{x})$. For a systematic catalogation of the symmetry groups of all the elements of $\Sigma_{NR}^{\star}$ I demand to the appendix L.1 "Chords and Third Chain Classes" of [9].

Given two tonalities $t_1 \, , \, t_2 \in \mathcal{T}$ such that $t_2 \in \mathcal{T}_{n.c.}(t_{1)}$, let's say $chord^{-1}(t_2) = \hat{T}_z \; chord^{-1}(t_2)$:

DEFINITION VIII.42

MAZZOLA MODULATOR FROM t_1 TO t_2:

a map $g : chord^{-1}(t_1) \rightarrow chord^{-1}(t_2)$ of the form:

$$g = T_z \circ h \; z \in \mathbb{Z}_{12} \, , \, h \in SYM(chord^{-1}(t_1)) \tag{8.99}$$

DEFINITION VIII.43

MAZZOLA MODULATION FROM t_1 TO t_2:

a couple $(g \, , \, c)$ such that:

1. g is a modulator from t_1 to t_2

2.
$$c \in \mathcal{C}[t_2 \, , \, \mathcal{T}_{n.c.}(t_2)] \tag{8.100}$$

where $\mathcal{T}_{n.c.}(t_2)$ is shortcut notation w.r.t. the definition VIII.39 of evident meaning.

DEFINITION VIII.44

MAZZOLA TONAL MUSICAL PIECES:

$$\mathcal{MP}_{Maz} := \{(hw_1\, m(t_1,t_2)\, hw_2 \cdots hw_{n-1}\, m(t_{n-1},t_n)\, hw_n) : \\ t_i \in \mathcal{T}_{Maz}\,, hw_i \in \mathcal{HW}(t_i)\,, m(t_{i-1},t_i) \in \mathcal{M}_{Maz}(t_{i-1},t_i)\, i = 1, \cdots, n \\ n \in \mathbb{N}\} \quad (8.101)$$

The Muzzulini-Mazzola's theory, introduces, at this point the notion of *modulation quantum* based on the analogy between the quanta mediating physical interaction in Particle Physics.

Such an analogy is, anyway, rather superficial since it is based simply on the visualization of Feynman diagram's in the perturbative expansions of Quantum Yang-Mills' Field Theories while the notion of symmetry, at the heart of the Mazzola-Muzzulini's modulation is in no way linked with some kind of "gauging" of a Lie group [84] [85], resulting in the symmetry group of the modulation quantum.

Let us observe, at this point, that, though explicitly thought to be the mathematical formalization of the three-stages' process through which Schönberg, in the 9^{th} chapter of [26], codifying the passage from a tonality t_1 to a tonality t_2 as:

1. an harmonic word in a tonality t_1
2. an harmonic word belonging both to t_1 and to t_2
3. a t_2-cadence determining the new tonality t_2

the Muzzulini-Mazzola's approach based on Symmetry Theory and r5esultin in the definitionVIII.43 imposes very stronger constraints.

Given two tonalities $t_1, t_2 \in \mathcal{T}$:

DEFINITION VIII.45

MODULATIONS FROM t_1 to t_2:

$$\mathcal{M}(t_1,t_2) = \{ (p,c) : p \in \mathcal{P}(t_1\,,\, t_2)\,,\, c \in \mathcal{C}[t_2\,,\, \mathcal{T}_{n.c.}(t_2)] \} \quad (8.102)$$

DEFINITION VIII.46

TONAL MUSICAL PIECES:

$$\mathcal{MP} := \{(hw_1\, m(t_1,t_2)\, hw_2 \cdots hw_{n-1}\, m(t_{n-1},t_n)\, hw_n) : t_i \in \mathcal{T}, hw_i \in \mathcal{HW}(t_i)\, m(t_{i-1},t_i) \in \mathcal{M}(t_{i-1},t_i)\, i = 1, \cdots, n \quad n \in \mathbb{N}\} \quad (8.103)$$

Example VIII.9

ITERATING THE II-V-I PROGRESSION MOVING ALONG THE FIFTH'S CYCLE

Though disliked in Classical Music, parallel motion of perfect fifth, and hence modulation of one step forward along the fifth's cycles, has been reconsidered in all kinds of Modern Music [82], [77], [78]. Such a kind of modulations are, in some way, the simplest ones since the tonalities connected by modulation have the highest number of pivotal degrees, though, as we have seen in the sectionVIII.5, such a number deacreases when the level at which tonalities are considered increases.

We have seen therein that the only pivotal degree persisting when the level of the major tonalities is raised from 1 to 4 is the 6^{th} degree of the departure-tonality coinciding with the 2^{th} degree of the arrival tonality. We shall therefore adopt such pivotal degree. As cadence we will adopt the celebrated cadential set (w.r.t. the natural context of the major tonalities) II-V-I progression.

We obtain conseguentially the following a tonal musical piece constisting in the parallel motion of the II-V-I progression along the fifths' cycle.

Adopting the expression implemented in the Mathematica notebook of section A:

```
stepalongfifthscycle[n_]:=Mod[n+7,12]

fifthscycle=NestList[stepalongfifthscycle,0,12];
```

the required musical piece may be immediately computed in the following way:

```
musicalpiece[n_,l_]:= Join[harmonicwordintermofdegrees[
      tonality[majorword[Part[fifthscycle,n]],l],{2,5,1}]]

Flatten[Table[musicalpiece[n,1],{n,1,13}],1] =
```

```
{{2,5,9},{7,11,2},{0,4,7},{9,0,4},{2,6,9},{7,11,2},{4,7,11},{9,1,4},{2,6,9},{
11,2,6},{4,8,11},{9,1,4},{6,9,1},{11,3,6},{4,8,11},{1,4,8},{6,10,1},{11,3,
6},{8,11,3},{1,5,8},{6,10,1},{3,6,10},{8,0,3},{1,5,8},{10,1,5},{3,7,10},{
    8,0,3},{5,8,0},{10,2,5},{3,7,10},{0,3,7},{5,9,0},{10,2,5},{7,10,2},{0,4,
    7},{5,9,0},{2,5,9},{7,11,2},{0,4,7}}

Flatten[Table[musicalpiece[n,2],{n,1,13}],1] =
{{2,5,9,0},{7,11,2,5},{0,4,7,11},{9,0,4,7},{2,6,9,0},{7,11,2,6},{4,7,11,2},{9,
1,4,7},{2,6,9,1},{11,2,6,9},{4,8,11,2},{9,1,4,8},{6,9,1,4},{11,3,6,9},{4,
    8,11,3},{1,4,8,11},{6,10,1,4},{11,3,6,10},{8,11,3,6},{1,5,8,11},{6,10,1,
5},{3,6,10,1},{8,0,3,6},{1,5,8,0},{10,1,5,8},{3,7,10,1},{8,0,3,7},{5,8,0,
3},{10,2,5,8},{3,7,10,2},{0,3,7,10},{5,9,0,3},{10,2,5,9},{7,10,2,5},{0,4,
    7,10},{5,9,0,4},{2,5,9,0},{7,11,2,5},{0,4,7,11}}

Flatten[Table[musicalpiece[n,3],{n,1,13}],1] =
{{2,5,9,0,4},{7,11,2,5,9},{0,4,7,11,2},{9,0,4,7,11},{2,6,9,0,4},{7,11,2,6,9},{
    4,7,11,2,6},{9,1,4,7,11},{2,6,9,1,4},{11,2,6,9,1},{4,8,11,2,6},{9,1,4,8,
11},{6,9,1,4,8},{11,3,6,9,1},{4,8,11,3,6},{1,4,8,11,3},{6,10,1,4,8},{11,3,
    6,10,1},{8,11,3,6,10},{1,5,8,11,3},{6,10,1,5,8},{3,6,10,1,5},{8,0,3,6,
10},{1,5,8,0,3},{10,1,5,8,0},{3,7,10,1,5},{8,0,3,7,10},{5,8,0,3,7},{10,2,
5,8,0},{3,7,10,2,5},{0,3,7,10,2},{5,9,0,3,7},{10,2,5,9,0},{7,10,2,5,9},{0,
    4,7,10,2},{5,9,0,4,7},{2,5,9,0,4},{7,11,2,5,9},{0,4,7,11,2}}

Flatten[Table[musicalpiece[n,4],{n,1,13}],1] =
```

```
{{2,5,9,0,4,7},{7,11,2,5,9,0},{0,4,7,11,2,5},{9,0,4,7,11,2},{2,6,9,0,4,7},{7,
11,2,6,9,0},{4,7,11,2,6,9},{9,1,4,7,11,2},{2,6,9,1,4,7},{11,2,6,9,1,4},{4,
8,11,2,6,9},{9,1,4,8,11,2},{6,9,1,4,8,11},{11,3,6,9,1,4},{4,8,11,3,6,9},{
1,4,8,11,3,6},{6,10,1,4,8,11},{11,3,6,10,1,4},{8,11,3,6,10,1},{1,5,8,11,3,
6},{6,10,1,5,8,11},{3,6,10,1,5,8},{8,0,3,6,10,1},{1,5,8,0,3,6},{10,1,5,8,
0,3},{3,7,10,1,5,8},{8,0,3,7,10,1},{5,8,0,3,7,10},{10,2,5,8,0,3},{3,7,10,
2,5,8},{0,3,7,10,2,5},{5,9,0,3,7,10},{10,2,5,9,0,3},{7,10,2,5,9,0},{0,4,7,
    10,2,5},{5,9,0,4,7,10},{2,5,9,0,4,7},{7,11,2,5,9,0},{0,4,7,11,2,5}}
```

Let us now analyze how the mathematical structure of Harmony is modified by giving up the constraint of the incommensurability of notes by replacing, as underlying starting scale, the equally-tempered chromatic scale $tempered_{eq}(C_2, 12)$ with the pytagoric scale $pytagoric(C_2)$.

I will here radically depart from Mazzola's way of treating both *pytagoric tuning* and *just-intonation tuning* whose philosophy could be described as hiding a problem by taking the quotient over that: the introduction of the quotient module J_{Kt} of the section 13.4.2.2 "Just Triadic Degree Interpretations" as well as that of the enharmonic projection enh of the section24.1.3 of [9] are therein functional to the concealment of a structrural problem inside an elegant mathematical abstraction.

But, please, tell a musician he have to play an equivalence class of notes differing by a comma and hear his answer $\cdots$ ....

The net effect of replacing the equally-tempered chromatic scale $tempered_{eq}(C_2, 12)$ with the pytagoric scale $pytagoric(C_2)$ may be formalized as the following ansatz concerning the adopted musical alphabet:

DEFINITION VIII.47

PYTAGORIC ANSATZ:

$$\mathbb{Z}_{12} \mapsto \mathbb{Z}_{12}^{\star} \tag{8.104}$$

where:

$$\mathbb{Z}_{12}^{\star} := \{[i]_{12}^{(n)} \, , \, i = 1, \cdots 12 \, , \, n \in \mathbb{N}\} \tag{8.105}$$

$$[i]_{12}^{(n)} := (ord((scale(\nu_{ref}, 3))_{12*n+1,12*n+11})_{i+1} \tag{8.106}$$

where the note $[i]_{12}^{(n)}$ is the new note obtained by $[i]_{12}$ after n turns of the *fifths' cycle* and where attention has to be kept as to the remarkVIII.1

The pytagoric ansatz may be also represented by the following change of musical notation :

$$C^{(n)} := [0]_{12}^{n} \tag{8.107}$$

$$C^{\sharp\,(n)} := [1]_{12}^{n} \tag{8.108}$$

$$D^{(n)} := [2]_{12}^{(n)} \tag{8.109}$$

$$D^{\sharp\,(n)} := [3]_{12}^{(n)} \tag{8.110}$$

$$E^{(n)} := [4]_{12}^{(n)} \tag{8.111}$$

$$F^{(n)} := [5]_{12}^{(n)} \tag{8.112}$$

$$F^{\sharp\,(n)} := [6]_{12}^{(n)} \tag{8.113}$$

$$G^{(n)} := [7]_{12}^{(n)} \tag{8.114}$$

$$G^{\sharp\,(n)} := [8]_{12}^{(n)} \tag{8.115}$$

$$A^{(n)} := [9]_{12}^{(n)} \tag{8.116}$$

$$A^{\sharp\,(n)} := [10]_{12}^{(n)} \tag{8.117}$$

$$B^{(n)} := [11]_{12}^{(n)} \tag{8.118}$$

$$\tag{8.119}$$

One has by construction that:

Theorem VIII.5

$$\mathbf{p}([i]_{12}^{(n+1)}) = \mathbf{p}([i]_{12}^{(n)}) + \mathbf{Kf} \;\; \forall n \in \mathbb{N} \, , \, i = 0, \cdots , 11 \tag{8.120}$$

Introduced the notation $\Xi := \mathbb{Z}_{12}^{\star}$:

DEFINITION VIII.48

FIFTH CYCLES' ORDERING RELATION OVER Ξ:

$$[n_1]_{12}^{(m)} \, <_{fc} \, [n_2]_{12}^{(m)} \, \Leftrightarrow \, n_1 \, < \, n_2 \tag{8.121}$$

$$[n_1]_{12}^{(m_1)} \, <_{fc} \, [n_2]_{12}^{(m_2)} \, \Leftrightarrow \, m_1 \, < \, m_2 \tag{8.122}$$

One has clearly that:

$$card(\Xi) \, = \, card(\mathbb{Z}_{12}^{\star}) \, = \, \aleph_0 \tag{8.123}$$

$$card(\Xi^{\star}) \, = \, card((\mathbb{Z}_{12}^{\star})^{\star}) \, = \, \aleph_1 \tag{8.124}$$

Given the pytagoric alphabet Ξ, a word $\tilde{x} \in \Xi^{\star}$ and a map $g \, : \, \Xi \rightarrow \Xi$:

DEFINITION VIII.49

MAP INDUCED BY g ON WORDS:

the map $\hat{g} \, : \, \Xi^{\star} \, \rightarrow \, \Xi^{\star}$:

$$\hat{g}(\tilde{x}) \, = \, \cdot_{i=1}^{|\tilde{x}|} g(x_i) \tag{8.125}$$

Given a letter $z \, := \, [i]_{12}^{(n)} \, \in \, \Xi$

DEFINITION VIII.50

PYTAGORIC TRANSLATION BY z :

the map $T_z \, : \, \Xi \, \mapsto \, \Xi$:

$$T_z([j]_{12}^{(m)}) \, = \, [i +_{12} j]^{(m)} \;\; \forall m \in \mathbb{N} \, , \, \forall j \in \{0, 1, \cdots, 11\} \tag{8.126}$$

DEFINITION VIII.51

PYTAGORIC INVERSION:

the map $Inv^{Pyt} \, : \, \Xi \, \mapsto \, \Xi$:

$$Inv([j]_{12}^{(m)}) \, = \, [0 \, -_{12} \, j]_{12}^{(m)} \;\; \forall m \in \mathbb{N} \, , \, \forall j \in \{0, 1, \cdots, 11\} \tag{8.127}$$

DEFINITION VIII.52

OPERATOR OF FIFTH CYCLE'S RAISING:

the map $C_{+} : \Xi \mapsto \Xi$:

$$C_{+}([j]_{12}^{(m)}) := [j]_{12}^{(m+1)} \quad \forall m \in \mathbb{N}, \forall j \in \{0, 1, \cdots, 11\} \tag{8.128}$$

DEFINITION VIII.53

OPERATOR OF FIFTH CYCLE'S LOWERING:

the map $C_{-} : \Xi \mapsto \Xi$:

$$C_{-}([j]_{12}^{(m)}) := [j]_{12}^{(m-1)} \quad \forall m \in \mathbb{N}, \forall j \in \{0, 1, \cdots, 11\} \tag{8.129}$$

Once again the effect of the pytagoric ansatz may be concretely analyzed adopting the following Mathematica expressions from the notebook of section A:

```
<<DiscreteMath‘Combinatorica‘;

<<DiscreteMath‘Permutations‘

putinorder[listofnumbers_]:=Permute[listofnumbers,Ordering[listofnumbers]]

pytagoricletter[n_,m_]:={letter[n],m}

pytagoricalphabetuptofifthcycles[m_]:=
  Flatten[Table[pytagoricletter[n,j],{j,0,m},{n,0,11}],1]

FROMpytagoricletterTOnote[x_]:=
  Part[putinorder[
      Take[scaleatfixedinterval[referencenote,3,12*Part[x,2]+11],-12]],
    Part[x,1]+1]

pytagoricscaleuptofifthcycles[m_]:=
  Map[FROMpytagoricletterTOnote,pytagoricalphabetuptofifthcycles[m]]
```

```
pytagoricword[w_,n_]:=Table[pytagoricletter[Part[w,i],n],{i,1,Length[w]}]
```

Example VIII.10

THE PART OF THE PYTAGORIC MUSICAL ALPHABET GENERATED BY THE FIRTS 10 FIFTHS' CYCLES

```
N[pytagoricscaleuptofifthcycles[10],225] =
```

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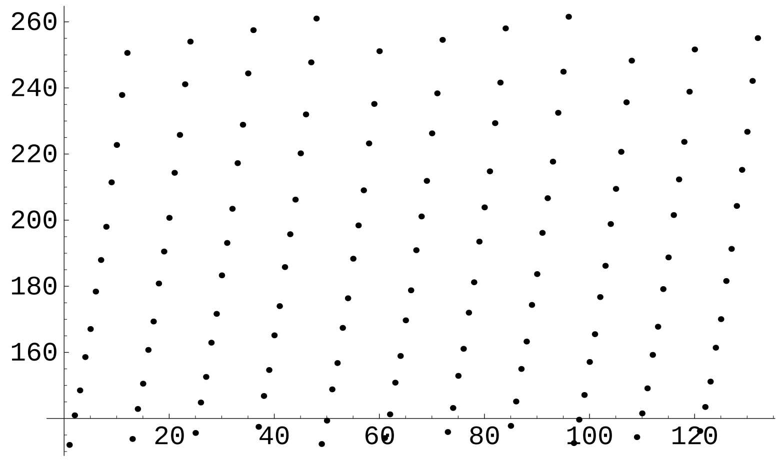
1333126150245789176551625132560729980468750000000000000000000000000000000}
```

(where we have shown to take such an high number of decimals to remark that only rational numbers are involved) that, graphically represented, appear as:

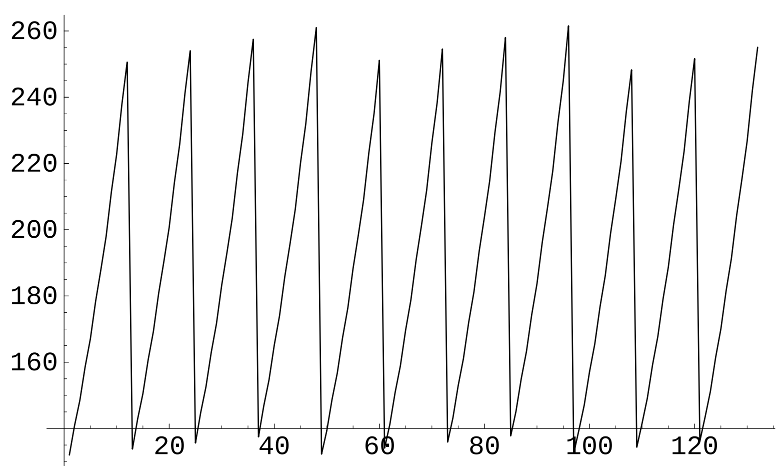
```
ListPlot[N[pytagoricscaleuptofifthcycles[10],225]] =
```



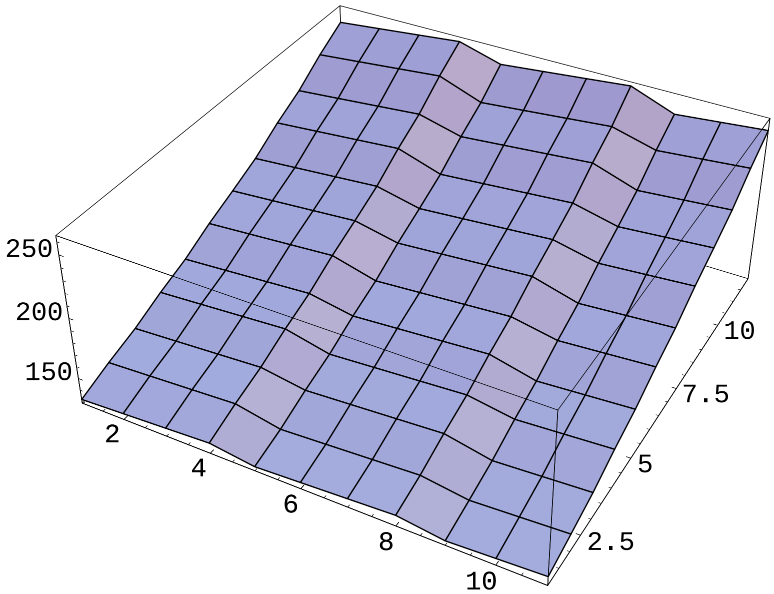
revealing, also in a pictorial way:

```
ListPlot[Table[

FROMpytagoricletterTOnote[Part[pytagoricalphabetuptofifthcycles[100],i]],{
      i,1,Length[pytagoricalphabetuptofifthcycles[100]]}],
  PlotJoined\[Rule]True] =
```




```
ListPlot3D[Table[FROMpytagoricletterTOnote[{i,n}],{i,0,11},{n,0,10}]]
=
```



the obvious fact that the ordering relation $<_{fc}$ over $\Xi^{\star}_{NR}$ doesn't correspond to the natural ordering relation $<$ among numbers

It is then interesting to see how all the previous formalization of Harmony is affected by the *pytagoric ansatz*;

given a word $\vec{x} = (x_1 , \cdots , x_{|\vec{x}|}) \in \mathbb{Z}^{\star}_{12}$ and a natural number $n \in \mathbb{N}$:

DEFINITION VIII.54

PYTAGORIC WORD OF $\vec{x}$ AT CYCLE n:

$$\tilde{pw}(\vec{x}, n) \ := (x_1^{(n)} , \cdots , x_{|\vec{x}|}^{(n)}) \tag{8.130}$$

Given a pytagoric word $\tilde{x} \in \Xi^{\star} - \bigcup_{k=1}^{4} \Xi^k$ and an integer $i \in \{1, \cdots , |\vec{x}|\}$:

DEFINITION VIII.55

MODE OF $\tilde{x}$ of i^{th} DEGREE :

$$mode(\tilde{x} , i) \ := \ \mathcal{S}^i_{cycl}(\tilde{x}) \tag{8.131}$$

where $\mathcal{S}_{cycl}$ denotes again the operator of cyclic shift.

Given furthermore an integer $n \in \mathbb{N}$

DEFINITION VIII.56

CHORD OF $\tilde{x}$ of i^{th} DEGREE AT LEVEL n :

$$chord(\tilde{x}\,,\,i\,,\,n)\ :=\ \cdot_{j=0}^{n+3} mode(\tilde{x}\,,\,i)_{2j+1} \tag{8.132}$$

where $\cdot$ denotes the concatenation operator while x_j denotes the j^{th} letter of the pytagoric word $\tilde{x}$.

Exactly as without the pytagoric ansatz, there exists a certain maximum level $maxlevel(\tilde{x})$ such that $chord(\tilde{x}\,,\,i\,,\,n)$ for $n > maxlevel(\tilde{x})$ simply adds notes already contained in the chord, again given by:

$$maxlevel(\tilde{x})\ =\ 1\ +\ Int(\frac{|\tilde{x}|-5}{2})\ -\ \frac{(-1)^{|\tilde{x}|}-1}{2}\dot{I}nt(\frac{|\tilde{x}|}{2}) \tag{8.133}$$

Let us now analyze how the previously discussed mathematical formalization of Harmony is affected by the pytagoric ansatz.

Given a pytagoric word $\tilde{x}\ \in\ \Xi_{NR}^{\star} - \bigcup_{k=1}^{4} \Xi_{NR}^{k}$ and a number $n \in \mathbb{N}_{+}\ :\ n < maxlevel(\vec{x})$:

DEFINITION VIII.57

PYTAGORIC TONALITY OF THE PYTAGORIC WORD $\tilde{x}\ \in\ \Xi_{NR}^{\star} - \bigcup_{k=1}^{4} \Xi_{NR}^{k}$ AT THE LEVEL n (n-TONALITY OF $\vec{x}$) :

$pytagorictonality[\tilde{x}\,,\,n]\ :=\ (\tilde{x}\,,\,chord^{(n)})$, where $chord^{(n)}\ :\ \{1,\cdots,|\tilde{x}|\}\ \mapsto\ \Xi^{\star}$ is the map such that:

$$chord^{(n)}(i)\ =\ chord(\tilde{x}\,,\,i\,,\,n) \tag{8.134}$$

I will denote the set of all the pytagoric n-tonalities by $\mathcal{T}_{n}^{Pyt}$ while I will denote by $\mathcal{T}^{Pyt}$ the set of all the pytagoric tonalities at any level.

One has clearly that:

$$card(\mathcal{T}_{n}^{Pyt})\ = card(\Xi_{NR}^{\star}\ -\ \bigcup_{k=1}^{4} \Xi_{NR}^{k})\ =\ \aleph_{1} \tag{8.135}$$

$$card(\mathcal{T})\ =\ card(\bigcup_{n \in \mathbb{N}} \mathcal{T}_{n}^{Pyt})\ =\ \aleph_{1} \tag{8.136}$$

Given a pytagoric n-tonality $t \in \mathcal{T}_{n}^{Pyt}$:

DEFINITION VIII.58

PYTAGORIC HARMONIC WORDS OF t:

$$\mathcal{HW}^{Pyt}(t) \ := \ (Range(chord^{(n)}(t))^{\star} \tag{8.137}$$

Given two pytagoric tonalities $t_1 \, , \, t_2 \ \in \ \mathcal{T}^{Pyt}$:

DEFINITION VIII.59

PYTAGORIC PIVOTAL DEGREES OF t_1 AND t_2:

$$\mathcal{P}^{Pyt}(t_1 \, , \, t_2) \ := \ Range(chord^{(level(t_1)}(t_1)) \bigcap Range(chord^{(level(t_2))}(t_2)) \tag{8.138}$$

Given a pytagoric tonality $t \in \mathcal{T}^{Pyt}$ and a context $\mathcal{T}^{Pyt}_{context} \subseteq \mathcal{T}^{Pyt}$

DEFINITION VIII.60

PYTAGORIC CADENCES OF t W.R.T. THE CONTEXT $\mathcal{T}_{context}$

$$\mathcal{C}^{Pyt}(t \, , \, \mathcal{T}_{context}) \ := \ \{c \ \in \ \mathcal{HW}^{Pyt}(t) \ : (\, c \ \in \ \mathcal{HW}^{Pyt}(t) \ \Rightarrow \ t \ = \ u \,) \ \forall u \in \mathcal{T}_{context}\} \tag{8.139}$$

Given a pytagoric word $\tilde{x} \in \Xi^{\star}$ and a natural number $n \in \mathbb{N} \, : \, n < maxlevel(\vec{x})$

DEFINITION VIII.61

PYTAGORIC NATURAL CONTEXT OF $pytagorictonality[\tilde{x}, n]$

$$\mathcal{T}^{Pyt}_{n.c.}(\tilde{x}, n) \ := \ \{pytagorictonality(\tilde{y} \, , \, n) \ : \ \tilde{x} \ \sim^{Pyt}_{T} \ \tilde{y}\} \tag{8.140}$$

where $\sim^{Pyt}_{T}$ is the pytagorical translational equivalence as induced by the map $\hat{T}^{Pyt}$ obtained applying the definitionVIII.49 to the alphabet Ξ.

DEFINITION VIII.62

PYTAGORIC MINIMAL CADENCES OF t W.R.T. THE CONTEXT $\mathcal{T}^{Pyt}_{context}$

$$\mathcal{MC}(t \, , \, \mathcal{T}^{Pyt}_{context}) \ := \ \{c_1 \ \in \ \mathcal{C}(t, \mathcal{T}_{context}) \ : \ \nexists c_2 \in \mathcal{C}(t, \mathcal{T}^{Pyt}_{context}), c_2 \ <_p \ c_1\} \tag{8.141}$$

Given two pytagoric tonalities $t_1, t_2 \in \mathcal{T}^{Pyt}$:

DEFINITION VIII.63

PYTAGORIC MODULATIONS FROM t_1 to t_2:

$$\mathcal{M}^{Pyt}(t_1, t_2) \;=\; \{\,(p, c)\,:\, p \in \mathcal{P}^{Pyt}(t_1\,,\,t_2)\,,\, c \in \mathcal{C}^{Pyt}(t_2)\,\} \tag{8.142}$$

DEFINITION VIII.64

PYTAGORIC TONAL MUSICAL PIECES:

$$\mathcal{MP}^{Pyt} \;:=\; \{(hw_1\, m(t_1, t_2)\, hw_2 \,\cdots\, hw_{n-1}\, m(t_{n-1}, t_n)\, hw_n)\;: \\ t_i \;\in\; \mathcal{T}^{Pyt}\,, hw_i \in \mathcal{HW}^{Pyt}(t_i)\, m(t_{i-1}, t_i) \;\in\; \mathcal{M}^{Pyt}(t_{i-1}, t_i)\, i = 1, \cdots, n \\ n \in \mathbb{N}\} \tag{8.143}$$

Remark VIII.3

PYTAGORICALLY-TUNED AND JUST-TUNED EULER NOTES VERSUS PYTAGORIC AND JUST SCALES

It is important, at this point to analyze the relation existing among the *pytagorical Euler notes* of definitionVI.9 and the *pytagoric scale pytagoric*$(\nu_{ref}, 3)$ as well as the relation existing among the *just-tuned Euler notes* of definitionVI.7 and the celebrated *just-intonation scale* [20] we are going to introduce.

Looking at the pitches obtained by pytagoric Euler points lying in a cube of side $= 10$ having as center the origin (i.e. the Euler point of ν_{ref}:

```
N[Map[FROMeulerpointTOpitch,Flatten[Table[x*octaveepoint+y*fifthpoint,

{x,-5,5},{y,-5,5}],1]]] =

{-15509.8,-13607.8,-11705.9,-9803.91,-7901.96,-6000.,-4098.04,-2196.09,-294.\
135,1607.82,
```

[20] often called the *simple-ratios' scale* or the *Zarlinian scale*, this last name being, anyway, based to an erroneous hystorical attribution to Gioseffo Zarlino (cfr. the voice "Scale Musicali (le) antiche e moderne" of [69]

```
3509.78,-14309.8,-12407.8,-10505.9,-8603.91,-6701.96,-4800.,-2898.04,-996.\
09,905.865,2807.82,

4709.78,-13109.8,-11207.8,-9305.87,-7403.91,-5501.96,-3600.,-1698.04,203.91,
  2105.87,4007.82,
  5909.78,-11909.8,-10007.8,-8105.87,-6203.91,-4301.96,-2400.,-498.045,
  1403.91,3305.87,5207.82,

7109.78,-10709.8,-8807.82,-6905.87,-5003.91,-3101.96,-1200.,701.955,2603.91,
  4505.87,6407.82,8309.78,-9509.78,-7607.82,-5705.87,-3803.91,-1901.96,0,
  1901.96,3803.91,5705.87,7607.82,

9509.78,-8309.78,-6407.82,-4505.87,-2603.91,-701.955,1200.,3101.96,5003.91,
  6905.87,8807.82,10709.8,-7109.78,-5207.82,-3305.87,-1403.91,498.045,2400.,

4301.96,6203.91,8105.87,10007.8,11909.8,-5909.78,-4007.82,-2105.87,-203.91,
  1698.04,3600.,5501.96,7403.91,9305.87,11207.8,
  13109.8,-4709.78,-2807.82,-905.865,996.09,2898.04,4800.,6701.96,8603.91,
  10505.9,12407.8,14309.8,-3509.78,-1607.82,294.135,2196.09,4098.04,6000.,
  7901.96,9803.91,11705.9,13607.8,15509.8}
```

and comparing it with the first 100 pitchs of the pytagoric scale:

```
N[Map[FROMnoteTOpitch,scaleatfixedinterval[132,3,100]]] =

{-15509.8,-13607.8,-11705.9,-9803.91,-7901.96,-6000.,-4098.04,-2196.09,-294.\
135,1607.82,

3509.78,-14309.8,-12407.8,-10505.9,-8603.91,-6701.96,-4800.,-2898.04,-996.\
09,905.865,2807.82,

4709.78,-13109.8,-11207.8,-9305.87,-7403.91,-5501.96,-3600.,-1698.04,203.91,
```

```
  2105.87,4007.82,
  5909.78,-11909.8,-10007.8,-8105.87,-6203.91,-4301.96,-2400.,-498.045,
  1403.91,3305.87,5207.82,

7109.78,-10709.8,-8807.82,-6905.87,-5003.91,-3101.96,-1200.,701.955,2603.91,
  4505.87,6407.82,8309.78,-9509.78,-7607.82,-5705.87,-3803.91,-1901.96,0,
  1901.96,3803.91,5705.87,7607.82,

9509.78,-8309.78,-6407.82,-4505.87,-2603.91,-701.955,1200.,3101.96,5003.91,
  6905.87,8807.82,10709.8,-7109.78,-5207.82,-3305.87,-1403.91,498.045,2400.,

4301.96,6203.91,8105.87,10007.8,11909.8,-5909.78,-4007.82,-2105.87,-203.91,
  1698.04,3600.,5501.96,7403.91,9305.87,11207.8,
  13109.8,-4709.78,-2807.82,-905.865,996.09,2898.04,4800.,6701.96,8603.91,
  10505.9,12407.8,14309.8,-3509.78,-1607.82,294.135,2196.09,4098.04,6000.,
  7901.96,9803.91,11705.9,13607.8,15509.8}
```

one is euristically led to discover that the *pytagoric Euler notes* and the *notes of the pytagoric scales* are different concepts:

```
  Intersection[
  N[Map[FROMeulerpointTOpitch,Flatten[Table[x*octaveepoint+y*fifthpoint
  ,{x,-5,5},{y,-5,5}],1]]],
N[Map[FROMnoteTOpitch,scaleatfixedinterval[132,3,100]]] =

  {0}
```

The situation is subtler as to *just intonation scales*:

DEFINITION VIII.65

DIATONIC JUST INTONATION SCALE OF C MAJOR:

the scale $\{\nu_1 , \cdots , \nu_7\}$ specified by the following table:

$\nu\, Hz$	$\mathbf{mi}(\nu, \nu_{ref})$	$\mathbf{p}(\nu)\, Cents$
132	1	0
148.5	$\frac{9}{8}$	203.91
165	$\frac{5}{4}$	386.314
176	$\frac{4}{3}$	498.045
198	$\frac{3}{2}$	701.955
220	$\frac{5}{3}$	719.354
247.5	$\frac{15}{8}$	1088.27

Behind the introduction of the scale of definitionVIII.65 laid a great musical innovation (that as many ones had to face the opposition of the Catholic Church) , namely the acceptance of the third-major interval as a consonant one.

Tonal Harmony was codified assuming the axiomVI.1 such an innovation assumed as cornerstone the answerV.1 to the question V.1.

Since:

- $harmonic(\nu_{ref}\,,\,1) \;=\; 2\nu_{ref}$ had been already used as a basis for the assumption of the axiomV.3 with the consequential constraint of the rescaling to $scale - range(\nu_{ref})$
- $harmonic(\nu_{ref}\,,\,2) \;=\; 3\nu_{ref}$ had been used in the construction of the pytagoric scale to fix the perfect fifth interval to the value $\mathbf{mi}(\nu_{perfect\,fifth}\,,\,\nu_{ref}) \;=\; \frac{3}{2}$
- $harmonic(\nu_{ref}\,,\,3) \;=\; 4\nu_{ref}$ couldn't add anything new w.r.t. the consequences induced by $harmonic(\nu_{ref}\,,\,1) \;=\; 2\nu_{ref}$

it was clear that the next protagonist had to be $harmonic(\nu_{ref}\,,\,2) \;=\; 5\nu_{ref}$.

One could, at this point, thought to introduce $scale(\nu_{ref}\,5)$ but this would have signified to throw away the consequences induced by $harmonic(\nu_{ref}\,,\,3)$.

Behind the introduction of the *just intonation diatonic scales* laid an other idea, namely the pytagorical creed that the musical consonant intervals correspond to simple ratios of integer numbers.

Given a rational number $r \;=\; \frac{n}{m}\, n, m \in \mathbb{N}\,,\, m \neq 0$ measures of its "simplicity" such as the following:

DEFINITION VIII.66

EMPIRICAL MEASURE OF SIMPLICITY OF r:

$$\mathcal{ESM}(r) \ := \ \frac{1}{gcd(n,m)} \frac{m+n}{m \cdot n} \tag{8.144}$$

(whose sometimes claimed link with physical consonance we confuted in the exampleVII.2) gave to the pytagorical major third intervals too low values:

$$\begin{aligned} \mathcal{ESM}(\mathbf{mi}(E^{(0)}, C^{(0)})) \ &= \ \mathcal{ESM}(\mathbf{mi}(A^{(0)}, F^{(0)})) \\ \mathcal{ESM}(\mathbf{mi}(B^{(0)}, G^{(0)})) \ &= \ \mathcal{ESM}(\frac{81}{64}) \ = \ \frac{145}{5184} \ = \ 0.0279707 \end{aligned} \tag{8.145}$$

specially if compared with:

$$\mathcal{ESM}(\frac{5}{4}) \ = \ \frac{9}{20} \ = \ 0.45 \tag{8.146}$$

The definitionVIII.65 was assumed as a sort of compromize between the necessity of implementing the new ratio $\frac{5}{4}$ for the third major's interval trying to conserve as much as possible the ratio $\frac{3}{2}$ for the perfect fifth's interval, a compromise lacking of the mathematical elegance of the pytagoric scale.

So, while preserving the ratio $\frac{3}{2}$ for the $C-G$, the $E-B$, $F-C'$ perfect fifths' intervals it gives a ratio of $\frac{40}{27}$ for the $D-A$ fifth interval for which:

$$\mathcal{ESM}(\frac{40}{27}) \ = \ \frac{67}{1080} \ = \ 0.062037 \tag{8.147}$$

Given the empirical nature of such a compromise, it doesn't astonish that the attempt of completing the diatonic just-intonation scale adding the 5 lacking notes necessary to obtain a putative chromatic just intonation scale following the same principle of looking for simple ratios, was performed by different authors in many ways.

Let us observe, at this point, that:

Theorem VIII.6

THE DIATONIC JUST-INTONATION MAJOR SCALE IS MADE OF JUST-EULER

NOTES:

$$\begin{aligned}
\mathcal{P}_{Euler}(\nu_{ref}) &= (\,0\,,\,0\,,\,0\,) && (8.148)\\
\mathcal{P}_{Euler}(\frac{9}{8}\nu_{ref}) &= (\,-3\,,\,2\,,\,0\,) && (8.149)\\
\mathcal{P}_{Euler}(\frac{5}{4}\nu_{ref}) &= (\,-2\,,\,0\,,\,1\,) && (8.150)\\
\mathcal{P}_{Euler}(\frac{4}{3}\nu_{ref}) &= (\,2\,,\,-1\,,\,0\,) && (8.151)\\
\mathcal{P}_{Euler}(\frac{3}{2}\nu_{ref}) &= (\,-1\,,\,1\,,\,0\,) && (8.152)\\
\mathcal{P}_{Euler}(\frac{5}{3}\nu_{ref}) &= (\,0\,,\,-1\,,\,1\,) && (8.153)\\
\mathcal{P}_{Euler}(\frac{15}{8}\nu_{ref}) &= (\,-3\,,\,1\,,\,1\,) && (8.154)\\
& && (8.155)
\end{aligned}$$

TheoremVIII.6 suggests to choice the other 5 notes among the simple-ratio's just Euler notes; one obtains in this way the following (cfr. the appendix K "Just and Well-Tempered Tuning" of [9]):

DEFINITION VIII.67

VOGEL'S CHROMATIC JUST-INTONATION SCALE OF C MAJOR

$$
\begin{aligned}
C^{(0)}_{just} &= \nu_{ref} && (8.156)\\
C^{\sharp\,(0)}_{just} &= \frac{16}{15}\nu_{ref} && (8.157)\\
D^{(0)}_{just} &= \frac{9}{8}\nu_{ref} && (8.158)\\
D^{\sharp\,(0)}_{just} &= \frac{6}{5}\nu_{ref} && (8.159)\\
E^{(0)}_{just} &= \frac{5}{4}\nu_{ref} && (8.160)\\
F^{(0)}_{just} &= \frac{4}{3}\nu_{ref} && (8.161)\\
F^{\sharp\,(0)}_{just} &= \frac{45}{32}\nu_{ref} && (8.162)\\
G^{(0)}_{just} &= \frac{3}{2}\nu_{ref} && (8.163)\\
G^{\sharp\,(0)}_{just} &= \frac{8}{5}\nu_{ref} && (8.164)\\
A^{(0)}_{just} &= \frac{5}{3}\nu_{ref} && (8.165)\\
A^{\sharp\,(0)}_{just} &= \frac{16}{9}\nu_{ref} && (8.166)\\
B^{(0)}_{just} &= \frac{15}{8}\nu_{ref} && (8.167)\\
& && (8.168)
\end{aligned}
$$

where:

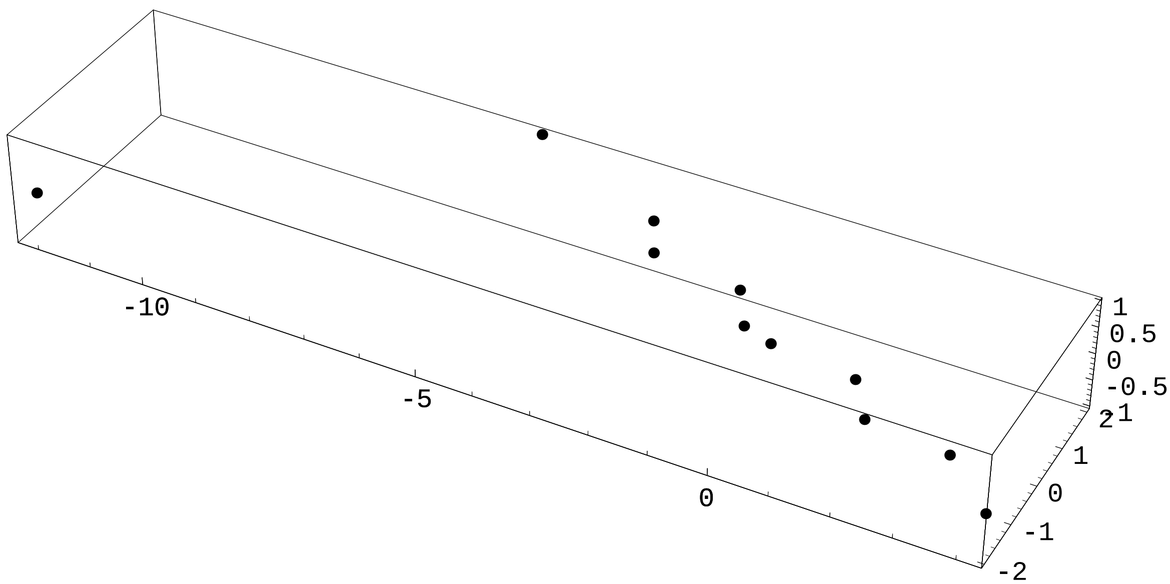
$$
\begin{aligned}
\mathcal{P}_{Euler}(C^{\sharp\,(0)}_{just}) &= (\,4\,,\,-1\,,\,-1\,)\\
\mathcal{P}_{Euler}(D^{\sharp\,(0)}_{just}) &= (\,1\,,\,1\,,\,-1\,)\\
\mathcal{P}_{Euler}(F^{\sharp\,(0)}_{just}) &= (\,-5\,,\,2\,,\,1\,)\\
\mathcal{P}_{Euler}(G^{\sharp\,(0)}_{just}) &= (\,3\,,\,0\,,\,-1\,)\\
\mathcal{P}_{Euler}(A^{\sharp\,(0)}_{just}) &= (\,4\,,\,-2\,,\,0\,)
\end{aligned}
$$

Visualized in Euler's space Vogel's chromatic just-intonation scale of C major appears as:

```
<<Graphics`Graphics3D`


vogelchromaticjustlistofeulerpoints:={{0,0,0},{4,-1,-1},{-3,2,0},{1,1,-1},
{-2,0,1},{2,-1,0},{-5,2,1},{-1,1,0},{3,0,-1},{0,-1,1},{4,-2,0}{-3,1,1}}
```

```
ScatterPlot3D[vogelchromaticjustlistofeulerpoints]
```



Let us, now, observe that whichever choice one effects as to the 5 chromatic notes one adopts to complete the diatonic just intonation scale of C major one have that, owing the the *third-comma*, fifth cycles don't close so that, a situation anbalogous to the one seen as to pytagoric scale occurs.

It follows that once again the passage from the 12-equally-tempered scale to the just-intonation scale may be formalized through the following:

DEFINITION VIII.68

JUST INTONATION ANSATZ:

$$\mathbb{Z}_{12} \; \mapsto \; \mathbb{Z}_{12}^{\star \, (just)} \tag{8.169}$$

where:

$$\mathbb{Z}_{12}^{\star \, (just)} \; := \; \{ [i]_{12}^{(n) \, (just} \, , \, i = 1, \cdots \, 12 \, , \, n \in \mathbb{N} \} \tag{8.170}$$

$$[i]_{12}^{(n)} \; := \; (ord((scale(\nu_{ref}, 3))_{12*n+1, 12*n+11})_{i+1} \tag{8.171}$$

where the note $[i]_{12}^{(n) \, (just)}$ is the new note obtained by $[i]_{12}$ after n turns of the *fifths' cycle*, defined by the conditions:

- $$\mathbf{p}([i]_{12}^{(n+1)\,(just)}) \; := \; \mathbf{p}([i]_{12}^{(n)\,(just)}) \, + \, \mathbf{Kt} \;\; \forall n \in \mathbb{N} \, , \, i = 0, \cdots , 11 \qquad (8.172)$$

- $$\mathcal{P}_{Euler}([i]_{12}^{(n)\,(just)}) \; \in \; \mathcal{N}_{Euler}^{just-tuned} \qquad (8.173)$$

The *just intonation ansatz* may be also represented by the following change of musical notation :

$$C^{(n)\,just} \; := \; [0]_{12}^{(n)\,just} \qquad (8.174)$$

$$C^{\sharp\,(n)\,just} \; := \; [1]_{12}^{(n)\,just} \qquad (8.175)$$

$$D^{(n)\,just} \; := \; [2]_{12}^{(n)\,just} \qquad (8.176)$$

$$D^{\sharp\,(n)\,just} \; := \; [3]_{12}^{(n)\,just} \qquad (8.177)$$

$$E^{(n)\,just} \; := \; [4]_{12}^{(n)\,just} \qquad (8.178)$$

$$F^{(n)\,just} \; := \; [5]_{12}^{(n)\,just} \qquad (8.179)$$

$$F^{\sharp\,(n)\,just} \; := \; [6]_{12}^{(n)\,just} \qquad (8.180)$$

$$G^{(n)\,just} \; := \; [7]_{12}^{(n)\,just} \qquad (8.181)$$

$$G^{\sharp\,(n)\,just} \; := \; [8]_{12}^{(n)\,just} \qquad (8.182)$$

$$A^{(n)\,just} \; := \; [9]_{12}^{(n)\,just} \qquad (8.183)$$

$$A^{\sharp\,(n)\,just} \; := \; [10]_{12}^{(n)\,just} \qquad (8.184)$$

$$B^{(n)\,just} \; := \; [11]_{12}^{(n)\,just} \qquad (8.185)$$

$$(8.186)$$

The effect of the *just-intonation ansatz* in the formalization of Harmony is structurally the same that the effect of the *pytagoric ansatz*. So it will be sufficient to repeat step by step the pytagoric footsteps to get the correct just-intonation notions in a straightforward way.

As to our goal, namely to show how these ansatz, required by the Vertical Rules of Tonal Harmony in order to obtain the physical consonance of notes, lead to an inconsistence in the formalization of Harmony, I will limit myself to take into account simply the pytagoric case, every consideration performed being trivially translated to the just-intonation case by replacing **Kf** with **Kt** etc.

In terms of the introduced mathematical staff our goal may be rephrased as the illustration of the deficiencies hidden inside $\mathcal{MP}^{Pyt}$ owing to the peculiarity of pytagoric Modulation's Theory

A first obvious consideration is codified by the following:

Theorem VIII.7

ON THE IMPOSSIBILITY OF MODULATING FROM ONE CYCLE TO ANOTHER

HP:

$\vec{x}\,,\,\vec{y} \in \mathbb{Z}_{12}^{\star}\ :\ \mathcal{M}(tonality[\vec{x},n], tonality[\vec{y},n]\ \neq\ \emptyset\ \ n \in \mathbb{N}\ :n\ \leq \min(maxlevel(\vec{x}), maxlevel(\vec{y}))$

TH:

$$\mathcal{M}^{Pyt}\{tonality[\tilde{pw}(\vec{x},i),n], tonality[\tilde{pw}(\vec{y},j),n]\}\ =\ \emptyset \forall i \neq j \tag{8.187}$$

PROOF:

Since:

$$\tilde{pw}(\vec{x},i) \bigcap \tilde{pw}(\vec{x},j)\ =\ \emptyset \tag{8.188}$$

it follows that:

$$\mathcal{P}^{Pyt}(\{tonality[\tilde{pw}(\vec{x},i),n], tonality[\tilde{pw}(\vec{y},j),n]\}\ =\ \emptyset \tag{8.189}$$

applying the definitionVIII.63 the thesis immediately follows

■

Example VIII.11

IMPOSSIBILITY OF MODULATING AMONG PYTAGORIC C MAJOR SCALES AT DIFFERENT CYCLES

```
tonality[pytagoricword[majorword[0],2],1] =

{{{0,2},{4,2},{7,2}},{{2,2},{5,2},{9,2}},{{4,2},{7,2},{11,2}},{{5,2},{9,2},{0,
      2}},{{7,2},{11,2},{2,2}},{{9,2},{0,2},{4,2}},{{11,2},{2,2},{5,2}}}

tonality[pytagoricword[majorword[0],3],1] =

{{{0,3},{4,3},{7,3}},{{2,3},{5,3},{9,3}},{{4,3},{7,3},{11,3}},{{5,3},{9,3},{0,
      3}},{{7,3},{11,3},{2,3}},{{9,3},{0,3},{4,3}},{{11,3},{2,3},{5,3}}}

Intersection[{{{0,2},{4,2},{7,2}},{{2,2},{5,2},{9,2}},{{4,2},{7,2},{11,2}},{{
5,2},{9,2},{0,2}},{{7,2},{11,2},{2,2}},{{9,2},{0,2},{4,2}},{{11,2},{2,
2},{5,2}}}
,{{{0,3},{4,3},{7,3}},{{2,3},{5,3},{9,3}},{{4,3},{7,3},{11,
3}},{{5,3},{9,3},{0,3}},{{7,3},{11,3},{2,3}},{{9,3},{0,3},{4,3}},{{11,
3},{2,3},{5,3}}} ] = {}
```

Let us then return to the Hindemith's citation presented in sectionIII, i.e.:

"Anyone who has ever tasted the delights of pure intonation by the continual displacement of the comma in string-quartet playing , must come to the conclusion that there can be no such thing as atonal music, in which the existence of tone relationship is denied" (cited from chapter 4 "Harmony", section 10 "Atonality and Polytonality" of [31])

and let us formalize Hindemith's "continual displacement of the comma" inside Pytagoric Modulation Theory.

Our key tools will be the fifth cycle's raising C_+ and lowering C_- operators of, respectively, definitionVIII.52 and definitionVIII.53 using them, first of all, to state the following obvious:

Corollary VIII.4

$$\mathcal{M}^{Pyt}\{tonality[\tilde{x},n],tonality[\hat{C}_+(\tilde{x}),n)]\} \;=\; \emptyset \;\; \forall \tilde{x} \in \Xi^{\star} \tag{8.190}$$

PROOF:

It is a particular case of theoremVIII.7 ■

Raising and lowering the letters of a pytagoric word in different ways one can result to pytagoric words whose tonality may be obtained by modulation by the starting one; I will call this kind of modulations *comma displacement modulations*.

To introduce them it is useful, first of all, to enlarge the mathematical toolbox in the following way: given a pytagoric word $\tilde{x} = (x_1 , \cdots , x_{|\vec{x}|}) \in \Xi^{\star}$

DEFINITION VIII.69

$$C_{\pm}(\tilde{x}, i) \ := \ \tilde{y} = (y_1 , \cdots , y_{|\vec{x}|}) \ : \ y_n = \begin{cases} C_{\pm} x_n, & \text{if } n = i; \\ x_n, & \text{otherwise.} \end{cases} \tag{8.191}$$

Obviously the operators of definitionVIII.69 easily implemented on computer through the following expressions from the notebook of sectionA:

```
cycleraising[pytagoricword_,i_]:=
  Table[If[n==i,{Part[pytagoricword,n,1],Part[pytagoricword,n,2]+1},
      Part[pytagoricword,n]],{n,1,Length[pytagoricword]}]


cyclelowering[pytagoricword_,i_]:=
  Table[If[n==i,{Part[pytagoricword,n,1],Part[pytagoricword,n,2]-1},
      Part[pytagoricword,n]],{n,1,Length[pytagoricword]}]
```

Given two pytagoric words $\tilde{x}$, $\tilde{y} \in \Xi^{\star}$:

DEFINITION VIII.70

$\tilde{x}$ IS A COMMA-DISPLACEMENT OF $\tilde{y}$ ($\tilde{x} \sim_K \tilde{y}$)

$\tilde{x}$ may be obtained from $\tilde{y}$ applying to it a suitable composition of the operators of definitionVIII.69

Given two pytagoric tonalities t_1 , $t_2 \in \mathcal{T}$:

DEFINITION VIII.71

t_1 IS A COMMA-DISPLACEMENT OF t_2 ($t_1 \sim_K t_2$):

$$\exists \tilde{x}_1, \tilde{x}_2 \in \Xi^{\star} \; : t_1 = tonality[\tilde{x}_1, n] \, , \, t_2 = tonality[\tilde{x}_2, n] \, , \tilde{x}_1 \sim_K \tilde{x}_2 \tag{8.192}$$

The notation adopted in definition VIII.70 and definitionVIII.71 is justified by the fact that in both cases the fact that comma displacement is an equivalence relation may be easily proved.

DEFINITION VIII.72

COMMA DISPLACEMENT'S PYTAGORIC MODULATIONS:

$$\mathcal{M}_K^{Pyt} \; := \; \{ m \in \mathcal{M}^{Pyt}(t_1, t_2) \, : \, t_1, t_2 \in \mathcal{T}^{Pyt} \, , \, t_1 \sim_K t_2 \} \tag{8.193}$$

Example VIII.12

A COMMA DISPLACEMENT MODULATION OF C MAJOR

Given the C major pytagoric word $\tilde{pw}_1$ of level zero:

```
pw1 = pytagoricword[majorword[0],0] =


{{0,0},{2,0},{4,0},{5,0},{7,0},{9,0},{11,0}}
```

and that, let's call it $\tilde{pw}_2$ obtained by $\tilde{pw}_1$ raising the 5^{th} letter:

```
pw2=cycleraising[pw1,5] =


{{0,0},{2,0},{4,0},{5,0},{7,1},{9,0},{11,0}}
```

let us introduce their tonalities of first level:

```
pt1=tonality[pw1,1] =


{{{0,0},{4,0},{7,0}},{{2,0},{5,0},{9,0}},{{4,0},{7,0},{11,0}},{{5,0},{9,0},{0,
      0}},{{7,0},{11,0},{2,0}},{{9,0},{0,0},{4,0}},{{11,0},{2,0},{5,0}}}


pt2=tonality[pw2,1] =


{{{0,0},{4,0},{7,1}},{{2,0},{5,0},{9,0}},{{4,0},{7,1},{11,0}},{{5,0},{9,0},{0,
      0}},{{7,1},{11,0},{2,0}},{{9,0},{0,0},{4,0}},{{11,0},{2,0},{5,0}}}
```

One has that:

```
pivotaldegreesintermofdegrees[pt1,pt2] =

{{2,4,6,7},{2,4,6,7}}
```

so, as to the passage in a neutral environment common to both the involved tonalities, there is no problem.

As to cadences we have, following the definition VIII.63, to consider as $\mathcal{T}_{context}$:

```
context =
Table[tonality[pytagoricword[majorword[n],0],1],{n,0,11}] =

{{{{0,0},{4,0},{7,0}},{{2,0},{5,0},{9,0}},{{4,0},{7,0},{11,0}},{{5,0},{9,0},{
        0,0}},{{7,0},{11,0},{2,0}},{{9,0},{0,0},{4,0}},{{11,0},{2,0},{5,

0}}},{{{1,0},{5,0},{8,0}},{{3,0},{6,0},{10,0}},{{5,0},{8,0},{0,0}},{{

6,0},{10,0},{1,0}},{{8,0},{0,0},{3,0}},{{10,0},{1,0},{5,0}},{{0,0},{3,

0},{6,0}}},{{{2,0},{6,0},{9,0}},{{4,0},{7,0},{11,0}},{{6,0},{9,0},{1,

0}},{{7,0},{11,0},{2,0}},{{9,0},{1,0},{4,0}},{{11,0},{2,0},{6,0}},{{1,

0},{4,0},{7,0}}},{{{3,0},{7,0},{10,0}},{{5,0},{8,0},{0,0}},{{7,0},{10,
        0},{2,0}},{{8,0},{0,0},{3,0}},{{10,0},{2,0},{5,0}},{{0,0},{3,0},{7,

0}},{{2,0},{5,0},{8,0}}},{{{4,0},{8,0},{11,0}},{{6,0},{9,0},{1,0}},{{

8,0},{11,0},{3,0}},{{9,0},{1,0},{4,0}},{{11,0},{3,0},{6,0}},{{1,0},{4,

0},{8,0}},{{3,0},{6,0},{9,0}}},{{{5,0},{9,0},{0,0}},{{7,0},{10,0},{2,
```

```
0}},{{9,0},{0,0},{4,0}},{{10,0},{2,0},{5,0}},{{0,0},{4,0},{7,0}},{{2,
        0},{5,0},{9,0}},{{4,0},{7,0},{10,0}}},{{{6,0},{10,0},{1,0}},{{8,0},{

11,0},{3,0}},{{10,0},{1,0},{5,0}},{{11,0},{3,0},{6,0}},{{1,0},{5,0},{
        8,0}},{{3,0},{6,0},{10,0}},{{5,0},{8,0},{11,0}}},{{{7,0},{11,0},{2,

0}},{{9,0},{0,0},{4,0}},{{11,0},{2,0},{6,0}},{{0,0},{4,0},{7,0}},{{2,

0},{6,0},{9,0}},{{4,0},{7,0},{11,0}},{{6,0},{9,0},{0,0}}},{{{8,0},{0,
        0},{3,0}},{{10,0},{1,0},{5,0}},{{0,0},{3,0},{7,0}},{{1,0},{5,0},{8,

0}},{{3,0},{7,0},{10,0}},{{5,0},{8,0},{0,0}},{{7,0},{10,0},{1,0}}},{{{

9,0},{1,0},{4,0}},{{11,0},{2,0},{6,0}},{{1,0},{4,0},{8,0}},{{2,0},{6,
        0},{9,0}},{{4,0},{8,0},{11,0}},{{6,0},{9,0},{1,0}},{{8,0},{11,0},{2,

0}}},{{{10,0},{2,0},{5,0}},{{0,0},{3,0},{7,0}},{{2,0},{5,0},{9,0}},{{

3,0},{7,0},{10,0}},{{5,0},{9,0},{0,0}},{{7,0},{10,0},{2,0}},{{9,0},{0,

0},{3,0}}},{{{11,0},{3,0},{6,0}},{{1,0},{4,0},{8,0}},{{3,0},{6,0},{10,

0}},{{4,0},{8,0},{11,0}},{{6,0},{10,0},{1,0}},{{8,0},{11,0},{3,0}},{{
        10,0},{1,0},{4,0}}}}
```

respect to which the pytagoric tonality pt_2 seems not to have has cadences:

```
cadencesintermofdegrees[pt2,3,context] =


{}
```

So is seems to be impossible to comma-modulate from pt1 to pt2.

An analogous situation occurs if $p\tilde{w}_2$ is defined as:

$$p\tilde{w}_2 \ := \ C_{+}(p\tilde{w}_1 \, , \, i) \ \ i = 1,2,3,4 \tag{8.194}$$

The situation is, anyway, different for $i = 6$:

```
pw2=cycleraising[pw1,6] =
{{0,0},{2,0},{4,0},{5,0},{7,0},{9,1},{11,0}}

pt2 =tonality[pw2,1] =
{{{0,0},{4,0},{7,0}},{{2,0},{5,0},{9,1}},{{4,0},{7,0},{11,0}},{{5,0},{9,1},{0,
      0}},{{7,0},{11,0},{2,0}},{{9,1},{0,0},{4,0}},{{11,0},{2,0},{5,0}}}

pivotaldegreesintermofdegrees[pt1,pt2] = {{1,3,5,7},{1,3,5,7}}

cadencesintermofdegrees[pt2,2,context] =
{{7},{1,7},{3,7},{5,7},{7,1},{7,3},{7,5},{7,7}}
```

so that $\{\{\{11,0\},\{2,0\},\{5,0\}\},\{\{11,0\},\{2,0\},\{5,0\}\}\} \in \mathcal{M}_{K}^{Pyt}$.

For i=7, anyway comma-modulation return to be impossible:

```
pw2=cycleraising[pw1,7] =
{{0,0},{2,0},{4,0},{5,0},{7,0},{9,0},{11,1}}

pivotaldegreesintermofdegrees[pt1,pt2] =
{{1,2,4,6},{1,2,4,6}}

cadences[pt2,2,context] = {}
```

The possibility of performing the Hindemith's comma-displacement not only in terms of the melody but at an harmonically structurally consistent way occurs, as we have empirically evidentiated, with very low frequency.

Remark VIII.4

THE STRUCTURAL INCONSISTENCE OF TONAL HARMONY VERSUS MEYER-EPPLER'S VALENCE THEORY

Since $\mathbb{Q}$ is dense in $\mathbb{R}$:

$$card(\{x \in (\mathbb{R} - \mathbb{Q}) \, : \, r_1 < x < r_2\} \; = \; \aleph_1 \;\; \forall r_1 \, , \, r_2 \in \mathbb{Q}, r_1 < r_2 \tag{8.195}$$

one could be led to think that the structural problem shown in this work is silly at on a physical ground in that, concretelly, Phychoacustics have tought to us that human hear cannot distinguish notes ω and $\omega + \epsilon$ nearer that a certain bound *epsilon*.

Such a phenomenon has been formalized by Werner Meyer-Eppler though his Valence Theory based on the following (cfr. the section B.2. " Discriminating Tones: Werner Meyer Eppler's Valence Theory" of [9]):

given two *sonor signals* s_1 and s_2 and a predicate P about sounds:

DEFINITION VIII.73

s_1 IS METAMERE TO s_2 W.R.T. P ($s_1 \sim_P s_2$)

s_1 cannot be distinguished by s_2 w.r.t. to P by human listeners.

DEFINITION VIII.74

VALENCE OF s_1 W.R.T. P (P-VALENCE OF s_1)

$$\sim_P s_1 \; := \; \{s_2 \, : \, s_1 \sim_P s_2\} \tag{8.196}$$

Identified a suitable set of predicates $\{P_i\}$ relevant as to sonor signal distinguishing, one considers as basic theoretical objects of Acoustics the valences w.r.t. to $\bigwedge_i P_i$

In particular it may be proved that infinite just-intonation Euler notes belong to the same $\bigwedge_i P_i$-valence.

As correctly stressed by Mazzola, the key bug of Meyer-Eppler's Valence Theory consists in that the relation $\sim_P$ is not an equivalence relation lacking both of the reflexive and, the more important point, of the transitive property.

In a less general context in which only *sounds* are taken into account and considering the valence of them with respect to the predicate

$$P(\nu) \; := << \; \nu \in (\nu - \frac{\epsilon}{2} \, , \, \nu + \frac{\epsilon}{2}) \; >> \tag{8.197}$$

one could think that, making in the definitionVII.1, definitionVII.3 and definition VII.6, the following:

DEFINITION VIII.75

P-VALENCE ANSATZ:

$$\delta_{Kronecker}(\sum_{i=1}^{N} n_i \omega_i) \ \rightarrow \ \delta^{\epsilon}_{Kronecker}(\sum_{i=1}^{N} n_i \omega_i) \tag{8.198}$$

where:

$$\delta^{\epsilon}_{Kronecker}(\omega) \ := \ \begin{cases} 1, \ \text{if } |\omega| \leq \epsilon \\ 0, \ \text{otherwise} \end{cases} \tag{8.199}$$

any claimed formal inconsistence of Tonal Harmony disappears.

This is not true since the structural problems concerning the extreme difficulty of making pytagoric modulations ,and specifically comma-modulation since any pytagoric modulation may be though as the composition of an "ordinary" modulation followed by a comma-modulation), that I have shown in the simple example VIII.12 are the effect of a structural problem.

And iterating the comma-displacements, for the level of degree that it is possible not only at a melodic but at an harmonic level, differences lying inside the same P-valence accumulate producing disasters as the (eventual) readers can concretelly compute using the notebook of A

APPENDIX A: THE MATHEMATICA'S NOTEBOOK *MATHEMATICAL MUSIC TOOLKIT*

```
Mathematical Music Toolkit (\[Alpha]-version - 8/2/2004 at 18:10)
In[1]:= Off[General::spell1]; In[2]:=
<<DiscreteMath`Combinatorica`; In[3]:=
<<DiscreteMath`Permutations` In[4]:=
<<DiscreteMath`KroneckerDelta` In[5]:=
putinorder[listofnumbers_]:=Permute[listofnumbers,Ordering[listofnumbers]]
In[6]:= \!\(\( (*\(**\ "\<generalizedselect[l,predicate,y]\>"\)\
picks\ out\ all\
      elements\ e\_i\ of\ the\ list\ l\ for\ which\ the\ binary\ predicate\
      predicate[e\_i, y]\ is\ True\ ***) \n (*
    \(**\ N\) . B .
      \( : \ since\ the\ instruction\ is\ implemented\ throwing\ away\
        elements\ of\ pattern\ String\ \ the\ list\ l\ must\ not\ to\
contain
        \ \(\(strings\ !!\)!\)\)\ ***) \ \)\)
In[7]:= generalizedselect[l_,predicate_,y_]:=
  DeleteCases[
    Table[If[predicate[Part[l,i],y],Part[l,i],"throw
away"],{i,1,Length[l]}],
    x_String]
In[8]:= (*** harmonic[nu,n] gives the n-th harmonic of the note nu
***) In[9]:= harmonic[nu_,n_]:=(n+1)*nu In[10]:= (***
scalerange[nu] gives the scale-range of the note nu ***) In[11]:=
scalerange[nu_]:=Interval[{nu,harmonic[nu,1]}] In[12]:= (***
rescalingtororange[nu,mu] rescales the note mu to the scale-
    range of the note nu ***)
In[13]:= \!\(rescalingtorange[nu_, mu_] := \
    If[mu\  < \ nu, \n\ \ \ \
```

```
      mu*\ 2\^IntegerPart[\(Log[nu] - Log[mu]\)\/Log[2] + 1], \
      If[mu > 2  nu, \nmu*2\^\(-IntegerPart[\(Log[mu] -
Log[nu]\)\/Log[2]]\),
        \n"\<undefined\>"]]\)
In[14]:= multiplication[nu_,ratio_]:=ratio*nu In[15]:=
multiplicationfornesting[list_]:={multiplication[Part[list,1],Part[list,2]],
    Part[list,2]}
In[16]:= scaleatfixedinterval[nu_,ratio_,n_]:=
  Prepend[Table[
      rescalingtorange[nu,
        Part[Nest[multiplicationfornesting,{nu,ratio},i],1]],{i,1,n}],nu]
In[17]:= orderedscaleatfixedinterval[nu_,ratio_,n_]:=
  putinorder[scaleatfixedinterval[nu,ratio,n]]
In[18]:= (*** referencenote is the absolute reference note as
given by a suitable
    diapason ***)
In[19]:= referencenote= 132; In[20]:= \!\( (*\(**\ the\)\
following\ instructions\ implements\ the\ musical\
      alphabet\ Z\_12\ ***) \)
In[21]:= letter[n_]:=Mod[n,12] In[22]:=
alphabet=Table[letter[n],{n,0,11}]; In[23]:= \!\(\( (*\(**\ the\)\
following\ instructions\ introduce\ the\ conversion\
        among\ the\ the\ musical\ alphabet\ Z\_12\ and\ the\ notes\ of\ the\
        12 - equally\ temmpered\ chromatic\ scale\ of\ referencenote; \
    words\ in\ Z\_12\ corresponds\ to\ Allen\ Forte' s\ notion\ of\ pitch -
      class - set\ ***) \ \ \)\)
In[24]:= \!\(FROMletterTOnote[n_] :=
referencenote*2\^\(letter[n]\/12\)\) In[25]:=
FROMwordTOscale[word_]:=
  Table[FROMletterTOnote[Part[word,i]],{i,1,Length[word]}]
In[26]:= (*** a musical piece is a list each element of which is
itself a list of two
```

```
    elemente: as note and its duration ***)
In[27]:=
FROMscaleTOpiece[scale_]:=Table[{Part[scale,i],crotchet},{i,1,Length[scale]}]
In[28]:=
FROMpieceTOscale[piece_]:=Table[Part[piece,i,1],{i,1,Length[piece]}]
In[29]:= (*** the following instructions implement the two
operation of translation
    and inversion at the basis of atonal music as well as the relative
    tests ***)
In[30]:=
translation[word_,n_]:=Table[Mod[Part[word,i]+n,12],{i,1,Length[word]}]
In[31]:= translationequivalenceofwordsQ[w1_,w2_]:=
  Not[Equal[Table[Equal[w2,translation[w1,n]],{n,0,11}],Table[False,{12}]]]
In[32]:=
inversion[word_]:=Table[Mod[12-Part[word,i],12],{i,1,Length[word]}]
In[33]:=
inversionequivalenceofwordsQ[w1_,w2_]:=Equal[w1,inversion[w2]]
In[34]:= inversioninvarianceofawordQ[w_]:=Equal[w,inversion[w]]
In[35]:= \!\(\( (*\(**\ mode[word, i]\)\ gives\ the\ i -
      th\ mode\ of\ a\ words\ on\ the\ musical\ alphabet\ \ \ Z\_12\ ***) \
  \)\)
In[36]:= mode[word_,i_]:=RotateLeft[word,i-1]; In[37]:= (*** the
following instructions introduces the most used words in Music
***) In[38]:=
majorword[0]:={letter[0],letter[2],letter[4],letter[5],letter[7],letter[9],
    letter[11]}
In[39]:=
majorword[n_]:=If[n==0,majorword[0],translation[majorword[0],n]]
In[40]:=
minorword[0]:={letter[0],letter[2],letter[3],letter[5],letter[7],letter[8],
    letter[10]}
In[41]:=
```

```
minorword[n_]:=If[n==0,minorword[0],translation[minorword[0],n]]
In[42]:=
harmonicminorword[0]:={letter[0],letter[2],letter[3],letter[5],letter[7],
    letter[8],letter[11]}
In[43]:= harmonicminorword[n_]:=
  If[n==0,harmonicminorword[0],translation[harmonicminorword[0],n]]
In[44]:=
dorianword[0]:=Mod[translation[mode[majorword[0],2],-2],12]
In[45]:=
dorianword[n_]:=If[n==0,dorianword[0],translation[dorianword[0],n]]
In[46]:=
phrigianword[0]:=Mod[translation[mode[majorword[0],3],-4],12]
In[47]:=
phrigianword[n_]:=If[n==0,phrigianword[0],translation[phrigianword[0],n]]
In[48]:=
lydianword[0]:=Mod[translation[mode[majorword[0],4],-5],12]
In[49]:=
lydianword[n_]:=If[n==0,lydianword[0],translation[lydianword[0],n]]
In[50]:=
mixolydianword[0]:=Mod[translation[mode[majorword[0],5],-7],12]
In[51]:= mixolydianword[n_]:=
  If[n==0,mixolydianword[0],translation[mixolydianword[0],n]]
In[52]:=
locrianword[0]:=Mod[translation[mode[majorword[0],7],-11],12]
In[53]:=
locrianword[n_]:=If[n==0,locrianword[0],translation[locrianword[0],n]]
In[54]:=
tziganminorword[0]:={letter[0],letter[2],letter[3],letter[6],letter[7],
    letter[8],letter[11]}
In[55]:= tziganminorword[n_]:=
  If[n==0,tziganminorword[0],translation[tziganminorword[0],n]]
In[56]:=
```

```
jewishword[0]:={letter[0],letter[1],letter[4],letter[5],letter[7],letter[8],
    letter[10]}
In[57]:=
jewishword[n_]:=If[n==0,jewishword[0],translation[jewishword[0],n]]
In[58]:=
indianword[0]:={letter[0],letter[1],letter[4],letter[5],letter[7],letter[8],
    letter[11]}
In[59]:=
indianword[n_]:=If[n==0,indianword[0],translation[indianword[0],n]]
In[60]:=
majorpentatonicword[0]:={letter[0],letter[2],letter[4],letter[7],letter[9]}
In[61]:= majorpentatonicword[n_]:=
  If[n==0,majorpentatonicword[0],translation[majorpentatonicword[0],n]]
In[62]:= minorpentatonicword[0]:=
  Mod[translation[mode[majorpentatonicword[0],6],-9] ,12]
In[63]:= minorpentatonicword[n_]:=
  If[n==0,minorpentatonicword[0],translation[minorpentatonicword[0],n]]
In[64]:=
bluesword[0]:={letter[0],letter[3],letter[5],letter[6],letter[7],letter[10]}
In[65]:=
bluesword[n_]:=If[n==0,bluesword[0],translation[bluesword[0],n]]
In[66]:=
esatonalword[0]:={letter[0],letter[2],letter[4],letter[6],letter[8],
    letter[10]}
In[67]:=
esatonalword[n_]:=If[n==0,esatonalword[0],translation[esatonalword[0],n]]
In[68]:=
augmentedword[0]:={letter[0],letter[3],letter[4],letter[7],letter[8],
    letter[11]}
In[69]:=
augmentedword[n_]:=If[n==0,augmentedword[0],translation[augmentedword[0],n]]
In[70]:=
```

```
halfwholediminishedword[0]:={letter[0],letter[1],letter[3],letter[4],
    letter[6],letter[7],letter[9],letter[10]}
In[71]:= halfwholediminishedword[n_]:=
  If[n==0,halfwholediminishedword[0],
    translation[halfwholediminishedword[0],n]]
In[72]:=
wholehalfdiminishedword[0]:={letter[0],letter[2],letter[3],letter[5],
    letter[6],letter[8],letter[9],letter[11]}
In[73]:= wholehalfdiminishedword[n_]:=
  If[n==0,wholehalfdiminishedword[0],
    translation[wholehalfdiminishedword[0],n]]
In[74]:=
wholetonediminishedword[0]:={letter[0],letter[1],letter[3],letter[4],
    letter[6],letter[8],letter[10]}
In[75]:= wholetonediminishedword[n_]:=
  If[n==0,wholetonediminishedword[0],
    translation[wholetonediminishedword[0],n]]
In[76]:=
bebopmajorword[0]:={letter[0],letter[2],letter[4],letter[5],letter[7],
    letter[8],letter[9],letter[11]}
In[77]:= bebopmajorword[n_]:=
  If[n==0,bebopmajorword[0],translation[bebopmajorword[0],n]]
In[78]:=
bebopdominant[0]:={letter[0],letter[2],letter[4],letter[5],letter[7],
    letter[9],letter[10],letter[11]}
In[79]:= bebopdominantword[n_]:=
  If[n==0,bebopdominantword[0],translation[bebopdominantword[0],n]]
In[80]:= chromaticword[0]:=alphabet In[81]:=
chromaticword[n_]:=If[n==0,chromaticword[0],translation[chromaticword[0],n]]
In[82]:= (*** randomword[n] gives a pseudo-
    random word on the musical alphabet of length n ***)
In[83]:= randomword[n_]:=Table[letter[ Random[Integer,{0,11}]
```

```
],{i,1,n}] In[84]:= (*** words[n] is the list of all the words of
length n in lexicographic
    ordering ***)
In[85]:= words[n_]:=Strings[chromaticword[0],n] In[86]:=
nonrepetitiveQ[l_]:=
  Equal[Table[
      If [Equal[Part[l,i],Part[l,j]],1,0],{i,1,Length[l]},{j,1,Length[l]}],
    IdentityMatrix[Length[l]]]
In[87]:= nonrepetitivewords[n_]:=Select[words[n],nonrepetitiveQ]
In[88]:= (*** "wordsupto[n]" is the list of all the words of
length less or equal to n
    in lexicographic ordering ***)
In[89]:=
wordsupto[n_]:=If[n==1,words[1],Join[wordsupto[n-1],words[n]]]
In[90]:= (*** "subwords[x]" gives the list of all the sub-
    words of the word x in lexicographic ordering ***)
In[91]:= subwords[x_]:=LexicographicSubsets[x] In[92]:= (***
"word[n]" gives the n^th word in  global lexicographic ordering
***) (*** N.B.: up to now it is defined only for n < 16 ***)
In[93]:= word[n_]:=Part[wordsupto[15],n] In[94]:= (***
"locallexicographicnumber[x]" gives the  local lexicographic
number of
    the word x,
  i.e. the lexicographic number of x w.r.t. to the set of the words having
    the same length of x ***)
In[95]:=
locallexicographicnumber[x_]:=Part[Flatten[Position[words[Length[x]],x]],1]
In[96]:= (*** "globallexicographicnumber[x]" gives the global
lexicographic number of
    the word x,
  i.e. the lexicographic number w.r.t. to the set of the words of any
    length   ***)
```

```
In[97]:= globallexicographicnumber[x_]:=
  If[Length[x]==1,locallexicographicnumber[x],
    Length[wordsupto[Length[x]-1]]+locallexicographicnumber[x]]
In[98]:= (*** "quasilexicographicnumber[x]" gives the
quasilexicographic number of
    the word x,
  i.e. the lexicographic number w.r.t. to the set of the words of any length
    considering also the null string  ***)
In[99]:=
quasilexicographicnumber[x_]:=If[x=={},0,globallexicographicnumber[x]+1]
In[100]:= (*** prefix[x,n] is the n-long prefix of the word x ***)
In[101]:= prefix[x_,n_]:=Part[x,Table[i,{i,1,n}]] In[102]:= (***
prefixes[x] is the list of the prefixes of the word x ***)
In[103]:= prefixes[x_]:=Table[prefix[x,n],{n,1,Length[x]-1}]
In[104]:= (*** prefixQ[x,y] is the predicate stating that x is a
prefix of y ***) In[105]:= prefixQ[x_,y_]:=MemberQ[prefixes[y],x]
In[106]:= (*** the following instructions introduced the notes
without the Mod-12
      constraint ***)
In[107]:= c[1]=FROMletterTOnote[0]; In[108]:= c[n_]:=2*c[n-1]
In[109]:= c\[Sharp][1]=FROMletterTOnote[1]; In[110]:=
c\[Sharp][n_]:=2*c\[Sharp][n-1] In[111]:=
d[1]=FROMletterTOnote[2]; In[112]:= d[n_]:=2*d[n-1] In[113]:=
d\[Sharp][1]=FROMletterTOnote[3]; In[114]:=
d\[Sharp][n_]:=2*d\[Sharp][n-1] In[115]:=
e[1]=FROMletterTOnote[4]; In[116]:= e[n_]:=2*e[n-1] In[117]:=
f[1]=FROMletterTOFROMletterTOnote[5]; In[118]:= f[n_]:=2*f[n-1]
In[119]:= f\[Sharp][1]=FROMletterTOnote[6]; In[120]:=
f\[Sharp][n_]:=2*f\[Sharp][n-1] In[121]:=
g[1]=FROMletterTOnote[7]; In[122]:= g[n_]:=2*g[n-1] In[123]:=
g\[Sharp][1]=FROMletterTOnote[8]; In[124]:=
g\[Sharp][n_]:=2*g\[Sharp][n-1] In[125]:=
```

```
a[1]=FROMletterTOnote[9]; In[126]:= a[n_]:=2*a[n-1] In[127]:=
a\[Sharp][1]=FROMletterTOnote[10]; In[128]:=
a\[Sharp][n_]:=2*a\[Sharp][n-1] In[129]:=
b[1]=FROMletterTOnote[11]; In[130]:= b[n_]:=2*b[n-1] In[131]:=
(*** insert the "referencetime" ***) In[132]:= referencetime=4;
In[133]:= semibreve=1*referencetime; In[134]:= minim=
(1/2)*referencetime; In[135]:= crotchet= (1/4)*referencetime;
In[136]:= quaver=(1/8)*referencetime; In[137]:=
semiquaver=(1/16)*referencetime; In[138]:=
demisemiquaver=(1/32)*referencetime; In[139]:=
hemidemisemiquaver=(1/64)*referencetime; In[140]:= (*** the
following instructions introduce the formalism of Euler's musical
    space ***)
In[141]:=
eulercoordination[x_List]:=
  Product[Power[Prime[i],Part[x,i]], {i,1,Length[x]}]
In[142]:= FROMeulerpointTOnote[eulpoint_List] :=
  referencenote*2^Part[eulpoint,1]*3^Part[eulpoint,2]*5^Part[eulpoint,3]
In[143]:= \!\(FROMnoteTOpitch[nu_] := 1200\/Log[2]*Log[nu\/\(\
referencenote\)\ ]\) In[144]:= \!\(FROMpitchTOnote[pitch_] := \ \
referencenote*Exp[Log[2]\/1200*pitch]\) In[145]:=
Hprime=Table[Log[Prime[i]],{i,1,3}]; In[146]:=
\!\(FROMeulerpointTOpitch[eulpoint_List] :=
    1200\/Log[2]*\ Dot[eulpoint, Hprime]\)
In[147]:=
lastpartisequaltosomethingQ[x_,something_]:=Equal[Last[x],something]
In[148]:= rationalsupto[n_]:=Flatten[
Table[Rational[i,j],{i,0,n},{j,1,n}] ] In[149]:=
FROMindexTOrational[index_,n_]:=Part[rationalsupto[n],index]
In[150]:= FROMlistofindicesTOlistofrationals[list_,n_]:=
  Table[FROMindexTOrational[Part[list,i],n],{i,1,Length[list]}]
In[151]:= FROMnoteTOeulerpoint[nu_,n_]:=
```

```
  FROMlistofindicesTOlistofrationals[
    First[generalizedselect[
        Flatten[Table[ {{i,j,k},
              Dot[{Part[ rationalsupto[2],i ],Part[ rationalsupto[2],j],
                  Part[ rationalsupto[2],k ]},Hprime]},{i,1,
              Length[ rationalsupto[2] ]},{j,1,Length[ rationalsupto[2]
]},{k,
              1,Length[ rationalsupto[2] ]}],2]  ,
        lastpartisequaltosomethingQ,FROMnoteTOpitch[nu]]],n]
In[152]:= FROMwordTOlistofeulerpoints[w_]:=
  Table[FROMnoteTOeulerpoint[Part[FROMwordTOscale[w] ,i]],{i,1,Length[w]}]
In[153]:= octaveepoint={1,0,0}; In[154]:= fifthpoint={0,1,0};
In[155]:= thirdpoint={0,0,1}; In[156]:=
canonicalnotesbasis:={octaveepoint,fifthpoint,thirdpoint}
In[157]:=
canonicalintervalsbasis:={octaveepoint,fifthpoint-octaveepoint,
    thirdpoint-2octaveepoint}
In[158]:= fifthcomma =-7*octaveepoint+ 12
*(fifthpoint-octaveepoint); In[159]:= thirdcomma=
  2*octaveepoint-4*(fifthpoint-octaveepoint)+thirdpoint-2octaveepoint;
In[160]:= gradussuavitatis[n_Integer]:=
  1+Sum[  ( FactorInteger[n][[i]][[1]]-1)* FactorInteger[n][[i]][[2]]  ,
{i,1,
        Length[FactorInteger[n]]}]
In[161]:= gradussuavitatis[r_Rational]:=
  gradussuavitatis[
    Times[Numerator[r]*Denominator[r],
      Power[GCD[Numerator[r],Power[Denominator[r],2]] , -1 ]]]
In[162]:= (***  chord[word,i,level] gives the chord of the i-
    th degree of a word at a chosen level ***)
In[163]:= chord[word_,i_,level_]:=
  Table[Part[mode[word,i],
```

```
If[Mod[2n+1,Length[word]]==0,Length[word],Mod[2n+1,Length[word]]]],{n,0,
      1+level}]
In[164]:= (*** numberoflevels of a word gives the possible chords
level of that
    word ***)
In[165]:= \!\(numbersoflevels[word_] :=
    1 + IntegerPart[\(Length[word] - 5\)\/2] -
      \(\((\(-1\))\)\^Length[word] - 1\)\/2*IntegerPart[Length[word]\/2]\)
In[166]:= (*** the following instructions introduce the basic
concept of Tonal
    Harmony ***)
In[167]:=
tonality[word_,level_]:=Table[chord[word,i,level],{i,1,Length[word]}]
In[168]:= pivotaldegrees[t1_,t2_]:=Intersection[t1,t2] In[169]:=
harmonicwords[t_,n_]:=Strings[t,n] In[170]:=
harmonicwordsupto[t_,n_]:=
  If[n==1,harmonicwords[t,1],
    Join[harmonicwordsupto[t,n-1],harmonicwords[t,n]]]
In[171]:= harmonicwordintermofdegrees[t_,listofdegrees_]:=
  Table[Part[t,Part[listofdegrees,i]],{i,1,Length[listofdegrees]}]
In[172]:=
degreeofachordinatonality[w_,t_]:=Part[Flatten[Position[t,w]],1]
In[173]:= (*** "pivotaldegreesintermofdegrees[t1,t2]" give a list
of two elements,
  the first being the pivotal degrees of "t1" and "t2" expressed as degrees
    of "t1", the second being the pivotal degrees of "t1" and "t2" expressed
    as degrees of t2" ***)
In[174]:= pivotaldegreesintermofdegrees[t1_,t2_]:={
    Table[degreeofachordinatonality[Part[pivotaldegrees[t1,t2],i],t1],{i,1,
        Length[pivotaldegrees[t1,t2]]}],
    Table[degreeofachordinatonality[Part[pivotaldegrees[t1,t2],i],t2],{i,1,
```

```
      Length[pivotaldegrees[t1,t2]]}]}
In[175]:= FROMharmonicwordTOphysicalchord[hw_]:=
  Table[FROMwordTOscale[Part[hw,i]],{i,1,Length[hw]}]
In[176]:=
tonalitymembershipQ[hw_,t_]:=MemberQ[harmonicwords[t,Length[hw]],hw]
In[177]:=
setofthemajortonalities[n_]:=Table[tonality[majorword[i],n],{i,0,11}];
In[178]:=
setoftheminortonalities[n_]:=Table[tonality[minorword[i],n],{i,0,11}];
In[179]:= setoftheclassicaltonalities[n_]:=
  Union[setofthemajortonalities[n],setoftheminortonalities[n]];
In[180]:= setofthegregoriantonalities[n_]:=
  Flatten[Table[tonality[mode[majorword[i],j],n],{j,1,7},{i,0,11}],1];
In[181]:= mazzolawords:=nonrepetitivewords[7] In[182]:=
mazzolaword[n_]:=Part[mazzolawords,n] In[183]:=
setofthemazzolatonalities[n_]:=

Table[tonality[Part[mazzolawords,i],n],{i,1,Length[nonrepetitivewords[7]]}];
In[184]:=
setofthejewishtonalities[n_]:=Table[tonality[jewishword[i],n],{i,0,11}];
In[185]:= truthQ[x_]:=Equal[x,True] In[186]:=
cadenceQ[hw_,setoftonalities_]:=
  If[Length[
      Select[Table[
          tonalitymembershipQ[hw,Part[setoftonalities,i]],{i,1,
            Length[setoftonalities]}],truthQ]]==1,True,False]
In[187]:= subharmonicwords[hw_]:=LexicographicSubsets[hw]
In[188]:= cadences[t_,n_,setoftonalities_]:=
  generalizedselect[harmonicwordsupto[t,n],cadenceQ,setoftonalities]
In[189]:= cadencesintermofdegrees[t_,n_,setoftonalities_]:=
  Table[degreeofachordinatonality[
     Part[Part[cadences[t,n,setoftonalities],i],j],t],{i,1,
```

```
      Length[cadences[t,n,setoftonalities] ]},{j,1,
      Length[ Part[cadences[t,n,setoftonalities],i] ]}]
In[190]:=
minimalcadenceQ[hw_,setoftonalities_]:=And[cadenceQ[hw,setoftonalities],

Equal[Table[cadenceQ[prefix[hw,i],setoftonalities],{i,1,Length[hw]-1}],
      Table[False,{Length[hw]-1}]]]
In[191]:= minimalcadences[t_,n_,setoftonalities_]:=
  generalizedselect[harmonicwords[t,n],minimalcadenceQ,setoftonalities]
In[192]:= minimalcadencesintermofdegrees[t_,n_,setoftonalities_]:=
  Table[degreeofachordinatonality[
      Part[Part[minimalcadences[t,n,setoftonalities],i],j],t],{i,1,
      Length[minimalcadences[t,n,setoftonalities] ]},{j,1,
      Length[ Part[minimalcadences[t,n,setoftonalities],i] ]}]
In[193]:= (*** a monodic piece is a list each element of which is
itself a list of the
    form {note ,duration}   ***)
In[194]:= playmonodic[piece_]:=
  Do[Play[Sin[piece[[i]][[1]] *2*\[Pi]*t],{t,0,piece[[i]][[2]]}] , {i,1,
      Length[piece]}]
In[195]:=
playwwordasarpeggio[w_]:=playmonodic[FROMscaleTOpiece[FROMwordTOscale[w]]]
In[196]:= playwordaschord[w_,time_]:=
  Play[Sum[Sin[Part[FROMletterTOnote[w],i]*2*\[Pi]*t],{i,1,Length[w]}],{t,0,
      time}]
In[197]:= playharmonicword[hw_,time_]:=
  Do[playwordaschord[Part[hw,i],time],{i,1,Length[hw]}]
In[198]:= (*** a polyphonic piece is a list each element of which
is itself a list of
    the form {word ,duration} ***)
In[199]:= playpolyphonic[piece_]:=
  Do[playwordaschord[piece[[i]][[1]],piece[[i]][[2]]] , {i,1,Length[piece]}]
```

```
In[200]:= translationofharmonicword[hw_,n_]:=
  Table[translation[Part[hw,i],n],{i,1,Length[hw]}]
In[201]:=
inversionofharmonicword[hw_]:=Table[inversion[Part[hw,i]],{i,1,Length[hw]}]
In[202]:= translationequivalenceofharmonicwordsQ[hw1_,hw2_]:=
  And[Length[hw1]==Length[hw2],
    Equal[Table[
        translationequivalenceofwordsQ[Part[hw1,i],Part[hw2,i]],{i,1,
          Length[hw1]}],Table[True,{Length[hw1]}]]]
In[203]:= inversionequivalenceofharmonicwordsQ[hw1_,hw2_]:=
  And[Length[hw1]==Length[hw2],
    Equal[Table[
        inversionequivalenceofwordsQ[Part[hw1,i],Part[hw2,i]],{i,1,
          Length[hw1]}],Table[True,{Length[hw1]}]]]
In[204]:= inversioninvarianceofanharmonicwordQ[hw_]:=
  Equal[Table[inversioninvarianceofawordQ[Part[hw,i]],{i,1,Length[hw]}],
    Table[True,{Length[w]}]]
In[205]:= stepalongfifthscycle[n_]:=Mod[n+7,12] In[206]:=
fifthscycle=NestList[stepalongfifthscycle,0,12]; In[207]:=
modulation[hw1_,hw2_,t1_,t2_,np_,cad2_,setoftonalities_]:=
  If[ Not[tonalitymembershipQ[hw1,t1]],False,
    If[ Not[tonalitymembershipQ[hw2,t2]],False,
      If[Not[MemberQ[pivotaldegrees[t1,t2],np ]],False,
        If[Not[MemberQ[cadences[t2,Length[np],setofonalities],cad2]],False,
          Join[hw1,np,cad2,hw2]]]]]
In[208]:= Mazzolamodulator[t1_,t2_]:=
  If[translationequivalenceofharmonicwordsQ[t1,t2],
    If[inversioninvarianceofanharmonicwordQ[t1],
      letter[n_]
        \[Rule]inversion[translation[ letter[n_] ,Part[t2,1]-Part[t1,1]]],
      letter[n_] \[Rule]translation[ letter[n_] ,Part[t2,1]-Part[t1,1]] ,
      "undefined"]]
```

```
In[209]:= MazzolamodulationQ[l_,t1_,t2_,n_,setoftonalities_]:=
  And[Equal[Part[l,1],Mazzolamodulator[t1,t2]],
    MemberQ[Part[l,2],mimimalcadentialsets[t,n,setoftonalities]]]
In[210]:= symmetrytransformation[g_]:=Rule[x_,g[x]] In[211]:=
symmetricQ[w_,g_]:=Equal[w,ReplaceAll[w,symmetrytransformation[g]]]
In[212]:= (*** an arbitarry law of harmony may be always codified
in terms of a
    predicate concerning tonalities and harmonic words ***)
In[213]:= lawofresolutiononthetonicQ[hw_,t_]:=
  And[tonalitymembershipQ[hw,t], Equal[Last[hw],Part[t,1]]]
In[214]:= specialtransformationrule[n_]:=
Rule[majorword[x_],majorword[x+n]] In[215]:=
specialinvarianceQ[n_,lawofharmomyQ_,argumentsofthelawofharmony__]:=
  Equal[lawofharmonyQ,
    ReplaceAll[lawofharmonyQ[argumentsofthelawofharmony],
      specialtransformationrule[n]]]
In[216]:= morethanspecialtransformationrule[n_]:=
Rule[mazzolaword[x_],mazzolaword[x+n]] In[217]:=
morethanspecialinvarianceQ[n_,lawofharmomyQ_,
argumentsofthelawofharmony__]:=
  Equal[lawofharmonyQ,
    ReplaceAll[lawofharmonyQ[argumentsofthelawofharmony],
      specialtransformationrule[n]]]
In[218]:= generaltransformationrule[y_scale]:=Rule[x_scale , y]
In[219]:=
generalinvariance[lawofharmonyQ,argumentsofthelawofharmony__,y_scale]:=
  Equal[x,ReplaceAll[lawofharmonyQ[argumentsofthelawofharmony],
      generaltransformationrule[y]]]
In[220]:= FROMnoteTOpulsation[nu_]:=Times[nu,Power[2*Pi,-1]]
In[221]:= FROMpulsationTOnote[omega_]:=2*Pi*omega In[222]:=
musicalinstrumentQ[a_]:=Sum[a[n]*a[-n],{n,-Infty,+Infty}]< + Infty
In[223]:= (*** a sound may be codified as a list of the form
```

```
{omega,a} where "omega" is
    a  pulsation,while "a" is a musical instrument ****)
In[224]:= FROMpulsationTOsound[omega_,instrument_]:=
  If[musicalinstrumentQ[instrument],{omega,instrument} , "undefined"]
In[225]:= FROMsoundTOpulsation[sound_]:=Part[sound,1] In[226]:=
physicalindexofconsonanceofsounds[sound1_,sound2_]:=
  Sum[KroneckerDelta[n*Part[sound1,1]+m*Part[sound2,1]]*(
        Part[sound1,2][n]+Part[sound2,2][m]),{n,-Infty,+Infty},{
      m,-Infty,+Infty}]
In[227]:=
physicalindexofconsonanceofpulsation[omega1_,omega2_,instrument_]:=
  If[musicalinstrumentQ[instrument],

physicalindexofconsonanceofsounds[FROMpulsationTOsound[omega1,instrument],
      FROMpulsationTOsound[omega2,instrument]]  , "undefined"]
In[228]:= commensurabilityQ[omega1_,omega2_]:=
  Equal[ Head[Times[omega1,Power[omega2,-1]]],Rational]
In[229]:= (*** the following instructions implement the pytagoric
ansatz ***) In[230]:= pytagoricletter[i_,n_]:={letter[i],n}
In[231]:= pytagoricalphabetuptofifthcycles[m_]:=
  Flatten[Table[pytagoricletter[n,j],{j,0,m},{n,0,11}],1]
In[232]:= FROMpytagoricletterTOnote[x_]:=
  Part[putinorder[
      Take[scaleatfixedinterval[referencenote,3,12*Part[x,2]+11],-12]],
    Part[x,1]+1]
In[233]:= pytagoricscaleuptofifthcycles[m_]:=
  Map[FROMpytagoricletterTOnote,pytagoricalphabetuptofifthcycles[m]]
In[234]:=
pytagoricword[w_,n_]:=Table[pytagoricletter[Part[w,i],n],{i,1,Length[w]}]
In[235]:= \!\(empiricalsimplicitymeasure[r_Rational] := \
    1\/GCD[Numerator[r], Denominator[r]]*
      \(Numerator[r] + Denominator[r]\)\/\(Numerator[r]*Denominator[r]\)\)
```

```
In[236]:=
vogelchromaticjustlistofeulerpoints:={{0,0,0},{4,-1,-1},{-3,2,0},{1,1,-1},{-2,
      0,1},{2,-1,0},{-5,2,1},{-1,1,0},{3,0,-1},{0,-1,1},{4,-2,0}{-3,1,1}}
In[237]:= cycleraising[pytagoricword_,i_]:=
  Table[If[n==i,{Part[pytagoricword,n,1],Part[pytagoricword,n,2]+1},
      Part[pytagoricword,n]],{n,1,Length[pytagoricword]}]
In[238]:= cyclelowering[pytagoricword_,i_]:=
  Table[If[n==i,{Part[pytagoricword,n,1],Part[pytagoricword,n,2]-1},
      Part[pytagoricword,n]],{n,1,Length[pytagoricword]}]
```

1. **Some useful palette**

Basic words' palette

```
(***********************************************************************

                    Mathematica-Compatible Notebook

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*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      4116,        117]*)
(*NotebookOutlinePosition[      5172,        156]*) (*
CellTagsIndexPosition[      5128,        152]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
        {
          ButtonBox[\(majorword[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(minorword[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(harmonicminorword[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(dorianword[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(phrigianword[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(lydianword[\[SelectionPlaceholder]]\)]},
```

```
{
  ButtonBox[\(mixolydianword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(locrianword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(tziganminorword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(jewishword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(majorpentatonicword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(minorpentatonicword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(bluesword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(esatonalword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(augmentedword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(halfwholediminishedword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(wholehalfdiminishedword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(wholetonediminishedword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(bebopmajorword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(bebopdominant[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(chromaticword[\[SelectionPlaceholder]]\)]},
{
  ButtonBox[\(randomword[\[SelectionPlaceholder]]\)]}
```

```
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      RowSpacings->0,
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  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  GroupPageBreakWithin->False,
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  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
  ScriptMinSize->9,
  ShowStringCharacters->True,
  NumberMarks->True,
  CounterIncrements->"Input",
  StyleMenuListing->None,
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}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
PageWidth->441, WindowSize->{Fit, Fit}, WindowMargins->{{60,
Automatic}, {Automatic, 33}}, WindowFrame->"Palette",
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WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
```

```
0}}, Magnification->1 ]


(*******************************************************************
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using Mathematica, you must remove the line containing CacheID at
the top of the file.  The cache data will then be recreated when
you save this file from within Mathematica.
*******************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 2402, 66, 362,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)




(*******************************************************************
End of Mathematica Notebook file.
*******************************************************************)
```

basic harmonic construction's palette

```
(***********************************************************************

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  email: info@wolfram.com
  phone: +1-217-398-0700 (U.S.)
```

```
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*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      8129,        231]*)
(*NotebookOutlinePosition[      9185,        270]*) (*
CellTagsIndexPosition[      9141,        266]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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            \(cadences[\[SelectionPlaceholder], \[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(words[\[SelectionPlaceholder]]\)],

          ButtonBox[
            \(cadencesintermofdegrees[\[SelectionPlaceholder],
              \[SelectionPlaceholder], \[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(wordsupto[\[SelectionPlaceholder]]\)],
```

```
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            \(minimalcadences[\[SelectionPlaceholder],
              \[SelectionPlaceholder], \[SelectionPlaceholder]]\)]},
        {
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          ButtonBox[
            \(minimalcadencesintermofdegrees[\[SelectionPlaceholder],
              \[SelectionPlaceholder], \[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
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              \)],

          ButtonBox[
            \(translationofharmonicword[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]},
        {
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          ButtonBox[\(inversionofharmonicword[\[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
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          ButtonBox[

\(translationequivalenceofharmonicwordsQ[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]},
        {
```

```
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      \[SelectionPlaceholder]]\)],

  ButtonBox[
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      \[SelectionPlaceholder]]\)]},
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  ButtonBox[
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      \)]},
{

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    \(Mazzolamodulator[\[SelectionPlaceholder],
      \[SelectionPlaceholder]]\)]},
{

  ButtonBox[
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      \[SelectionPlaceholder]]\)],

  ButtonBox[
    \(MazzolamodulationQ[\[SelectionPlaceholder],
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```

```
            \[SelectionPlaceholder], \[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
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          ButtonBox[
            \(modulation[\[SelectionPlaceholder], \[SelectionPlaceholder],
              \[SelectionPlaceholder], \[SelectionPlaceholder],
              \[SelectionPlaceholder], \[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
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\[SelectionPlaceholder]]
              \)],

          ButtonBox[
            \(minimalcadenceQ[\[SelectionPlaceholder],
              \[SelectionPlaceholder], \[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(harmonicwords[\[SelectionPlaceholder],
\[SelectionPlaceholder]]
              \)],
          ButtonBox[\({\[SelectionPlaceholder]}\)]},
        {

          ButtonBox[
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```
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        ButtonBox[\({\[SelectionPlaceholder],
\[SelectionPlaceholder]}\)]},
      {

        ButtonBox[
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        ButtonBox[
          \({\[SelectionPlaceholder], \[SelectionPlaceholder],
            \[SelectionPlaceholder]}\)]},
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            \[SelectionPlaceholder]]\)],

        ButtonBox[
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            \[SelectionPlaceholder], \[SelectionPlaceholder]}\)]},
      {

        ButtonBox[
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        ButtonBox[
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            \[SelectionPlaceholder]}\)]},
      {
```

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        ButtonBox[
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            \[SelectionPlaceholder]]\)],

        ButtonBox[
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            \[SelectionPlaceholder], \[SelectionPlaceholder],
            \[SelectionPlaceholder]}\)]}
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Evaluatable->True,
CellGroupingRules->"InputGrouping",
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DefaultFormatType->DefaultInputFormatType,
LineSpacing->{1.25, 0},
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  FormatType->InputForm,
  ScriptMinSize->9,
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  NumberMarks->True,
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  StyleMenuListing->None,
  FontFamily->"Courier",
  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
PageWidth->820, WindowSize->{Fit, Fit}, WindowMargins->{{30,
Automatic}, {Automatic, 48}}, WindowFrame->"Palette",
WindowElements->{}, WindowFrameElements->"CloseBox",
WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


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***********************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)
```

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(*NotebookFileOutline Notebook[{ Cell[1710, 49, 6415, 180, 330,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)



(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

basic set of tonalities' palette

```
(***********************************************************************

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  email: info@wolfram.com
  phone: +1-217-398-0700 (U.S.)

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*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      2997,         89]*)
(*NotebookOutlinePosition[      4053,        128]*) (*
CellTagsIndexPosition[      4009,        124]*)
(*WindowFrame->Palette*)
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        {

          ButtonBox[
            \(setoftheclassicaltonalities[\[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(setofthegregoriantonalities[\[SelectionPlaceholder]]\)]},
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ButtonBox[\(setofthemazzolatonalities[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(setofthejewishtonalities[\[SelectionPlaceholder]]\)]}
        },
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  CellGroupingRules->"InputGrouping",
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  PageBreakWithin->False,
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  CellLabelMargins->{{11, Inherited}, {Inherited, Inherited}},
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  FormatType->InputForm,
  ScriptMinSize->9,
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  StyleMenuListing->None,
  FontFamily->"Courier",
  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
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ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


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you save this file from within Mathematica.
*******************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)
```

```
(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1283, 38, 106,
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  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)



(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

play palette

```
(***********************************************************************

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```
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(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      2940,         89]*)
(*NotebookOutlinePosition[      3996,        128]*) (*
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CellTagsIndexPosition[      3952,        124]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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        {
          ButtonBox[\(playpolyphonic[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(playwwordasarpeggio[\[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(playwordaschord[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(playharmonicword[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]}
        },
      RowSpacings->0,
      ColumnSpacings->0,
      GridDefaultElement:>ButtonBox[ "\\[Placeholder]"]]], NotebookDefault,
  CellMargins->{{Inherited, Inherited}, {5, Inherited}},
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  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  GroupPageBreakWithin->False,
```

```
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  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
  ScriptMinSize->9,
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  NumberMarks->True,
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  StyleMenuListing->None,
  FontFamily->"Courier",
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}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
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Automatic}, {Automatic, 18}}, WindowFrame->"Palette",
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Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


(***********************************************************************
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you save this file from within Mathematica.
***********************************************************************)
```

```
(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1226, 38, 90,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)




(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

times'palette

```
(***********************************************************************

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(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      2744,         87]*)
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(*NotebookOutlinePosition[      3801,        126]*) (*
CellTagsIndexPosition[      3757,        122]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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        {
          ButtonBox["demisemiquaver"]},
        {
          ButtonBox["hemidemisemiquaver"]}
        },
      RowSpacings->0,
      ColumnSpacings->0,
      GridDefaultElement:>ButtonBox[ "\\[Placeholder]"]]], NotebookDefault,
  CellMargins->{{Inherited, Inherited}, {5, Inherited}},
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  GroupPageBreakWithin->False,
  CellLabelMargins->{{11, Inherited}, {Inherited, Inherited}},
```

```
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  FormatType->InputForm,
  ScriptMinSize->9,
  ShowStringCharacters->True,
  NumberMarks->True,
  CounterIncrements->"Input",
  StyleMenuListing->None,
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  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
PageWidth->384, WindowSize->{Fit, Fit}, WindowMargins->{{100,
Automatic}, {Automatic, 33}}, WindowFrame->"Palette",
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WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


(***********************************************************************
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the top of the file.  The cache data will then be recreated when
you save this file from within Mathematica.
***********************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)
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(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1030, 36, 122,
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  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)




(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

Euler space's palette

```
(***********************************************************************

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```
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(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      3619,        107]*)
(*NotebookOutlinePosition[      4675,        146]*) (*
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CellTagsIndexPosition[      4631,        142]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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          ButtonBox["fifthpoint"]},
        {
          ButtonBox[\(FROMeulerpointTOnote[\[SelectionPlaceholder]]\)],
          ButtonBox["thirdpoint"]},
        {
          ButtonBox[\(FROMnoteTOpitch[\[SelectionPlaceholder]]\)],
          ButtonBox["canonicalnotesbasis"]},
        {
          ButtonBox[\(FROMpitchTOnote[\[SelectionPlaceholder]]\)],
          ButtonBox["canonicalintervalsbasis"]},
        {
          ButtonBox[\(FROMeulerpointTOpitch[\[SelectionPlaceholder]]\)],
          ButtonBox["fifthcomma"]},
        {

          ButtonBox[
            \(FROMnoteTOeulerpoint[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)],
          ButtonBox["thirdcomma"]},
        {

          ButtonBox[
            \(FROMwordTOlistofeulerpoints[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)],
```

```
          ButtonBox[
            \(gradussuavitatis[\[SelectionPlaceholder]  _Integer]\)]},
        {
          ButtonBox["octaveepoint"],

          ButtonBox[
            \(gradussuavitatis[\[SelectionPlaceholder]  _Rational]\)]}
        },
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      ColumnSpacings->0,
      GridDefaultElement:>ButtonBox[ "\\[Placeholder]"]]], NotebookDefault,
  CellMargins->{{Inherited, Inherited}, {5, Inherited}},
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  GroupPageBreakWithin->False,
  CellLabelMargins->{{11, Inherited}, {Inherited, Inherited}},
  DefaultFormatType->DefaultInputFormatType,
  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
  ScriptMinSize->9,
  ShowStringCharacters->True,
  NumberMarks->True,
  CounterIncrements->"Input",
  StyleMenuListing->None,
  FontFamily->"Courier",
  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
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Automatic}, {Automatic, 18}}, WindowFrame->"Palette",
WindowElements->{}, WindowFrameElements->"CloseBox",
WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


(*******************************************************************
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using Mathematica, you must remove the line containing CacheID at
the top of the file.  The cache data will then be recreated when
you save this file from within Mathematica.
*******************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1905, 56, 142,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)
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(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

musical relativity's palette

```
(***********************************************************************

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```
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  email: info@wolfram.com
  phone: +1-217-398-0700 (U.S.)

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Wolfram Research.
*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      3513,        107]*)
(*NotebookOutlinePosition[      4569,        146]*) (*
CellTagsIndexPosition[      4525,        142]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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        {

          ButtonBox[
            \(symmetricQ[\[SelectionPlaceholder], \[SelectionPlaceholder]]
              \)]},
```

```
        {

          ButtonBox[
            \(lawofresolutiononthetonicQ[\[SelectionPlaceholder],
              \[SelectionPlaceholder]]\)]},
        {

ButtonBox[\(specialtransformationrule[\[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(specialinvarianceQ[\[SelectionPlaceholder],
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        {

          ButtonBox[

\(morethanspecialtransformationrule[\[SelectionPlaceholder]]\)]},
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          ButtonBox[
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  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
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  GroupPageBreakWithin->False,
  CellLabelMargins->{{11, Inherited}, {Inherited, Inherited}},
  DefaultFormatType->DefaultInputFormatType,
  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
  ScriptMinSize->9,
  ShowStringCharacters->True,
  NumberMarks->True,
  CounterIncrements->"Input",
  StyleMenuListing->None,
  FontFamily->"Courier",
  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
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Automatic}, {Automatic, 48}}, WindowFrame->"Palette",
WindowElements->{}, WindowFrameElements->"CloseBox",
WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]
```

```
(***********************************************************************
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you save this file from within Mathematica.
***********************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1799, 56, 172,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)



(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

monodic palette

```
(***********************************************************************
```

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  email: info@wolfram.com
  phone: +1-217-398-0700 (U.S.)

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Wolfram Research.
*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      8418,        189]*)
(*NotebookOutlinePosition[      9474,        228]*) (*
CellTagsIndexPosition[      9430,        224]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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```
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```

```
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```

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      GridDefaultElement:>ButtonBox[ "\\[Placeholder]"]]], NotebookDefault,
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  CellGroupingRules->"InputGrouping",
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  PageBreakWithin->False,
  GroupPageBreakWithin->False,
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  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
  ScriptMinSize->9,
  ShowStringCharacters->True,
  NumberMarks->True,
  CounterIncrements->"Input",
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  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
PageWidth->866, WindowSize->{Fit, Fit}, WindowMargins->{{10,
Automatic}, {Automatic, 18}}, WindowFrame->"Palette",
WindowElements->{}, WindowFrameElements->"CloseBox",
WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
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ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


(***********************************************************************
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the top of the file.  The cache data will then be recreated when
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***********************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 6704, 138, 218,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)




(***********************************************************************
End of Mathematica Notebook file.
***********************************************************************)
```

notes' palette

```
(***********************************************************************

                    Mathematica-Compatible Notebook

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*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      3654,        109]*)
(*NotebookOutlinePosition[      4709,        148]*) (*
CellTagsIndexPosition[      4665,        144]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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          ButtonBox[\(c[\[SelectionPlaceholder]]\)]},
        {
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          ButtonBox[\(c\[Sharp][\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(d[1]\)],
          ButtonBox[\(d[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(d\[Sharp][1]\)],
          ButtonBox[\(d\[Sharp][\[SelectionPlaceholder]]\)]},
```

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        ButtonBox[\(e[\[SelectionPlaceholder]]\)]},
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        ButtonBox[\(f[1]\)],
        ButtonBox[\(f[\[SelectionPlaceholder]]\)]},
      {
        ButtonBox[\(f\[Sharp][1]\)],
        ButtonBox[\(f\[Sharp][\[SelectionPlaceholder]]\)]},
      {
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        ButtonBox[\(g[\[SelectionPlaceholder]]\)]},
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        ButtonBox[\(g\[Sharp][1]\)],
        ButtonBox[\(g\[Sharp][\[SelectionPlaceholder]]\)]},
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        ButtonBox[\(a[\[SelectionPlaceholder]]\)]},
      {
        ButtonBox[\(a\[Sharp][1]\)],
        ButtonBox[\(a\[Sharp][\[SelectionPlaceholder]]\)]},
      {
        ButtonBox[\(b[1]\)],
        ButtonBox[\(b[\[SelectionPlaceholder]]\)]}
      },
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    ColumnSpacings->0,
    GridDefaultElement:>ButtonBox[ "\\[Placeholder]"]]], NotebookDefault,
CellMargins->{{Inherited, Inherited}, {5, Inherited}},
Evaluatable->True,
CellGroupingRules->"InputGrouping",
PageBreakAbove->True,
```

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  PageBreakWithin->False,
  GroupPageBreakWithin->False,
  CellLabelMargins->{{11, Inherited}, {Inherited, Inherited}},
  DefaultFormatType->DefaultInputFormatType,
  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
  ScriptMinSize->9,
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  StyleMenuListing->None,
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  FontWeight->"Bold"]
}, FrontEndVersion->"Microsoft Windows 3.0", ScreenRectangle->{{0,
1024}, {0, 712}}, Editable->False, WindowToolbars->{},
PageWidth->334, WindowSize->{Fit, Fit}, WindowMargins->{{80,
Automatic}, {Automatic, 3}}, WindowFrame->"Palette",
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WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


(************************************************************************
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```

```
*******************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1940, 58, 202,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  CounterIncrements->"Input"]
} ] *)




(*******************************************************************
End of Mathematica Notebook file.
*******************************************************************)
```

pytagoric palette

```
(*******************************************************************

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*******************************************************************)

(*CacheID: 232*)
```

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(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      3377,        105]*)
(*NotebookOutlinePosition[      4433,        144]*) (*
CellTagsIndexPosition[      4389,        140]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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          ButtonBox[
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              \[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(pytagoricalphabetuptofifthcycles[\[SelectionPlaceholder]]\)]},
        {

          ButtonBox[
            \(pytagoricscaleuptofifthcycles[\[SelectionPlaceholder]]\)]},
        {

ButtonBox[\(FROMpytagoricletterTOnote[\[SelectionPlaceholder]]\)]},
        {
          ButtonBox[\(FROMeulerpointTOpitch[fifthcomma]\)]},
        {

          ButtonBox[
            \(pytagoricword[\[SelectionPlaceholder],
\[SelectionPlaceholder]]
```

```
              \)]},
        {

          ButtonBox[
            \(cycleraising[\[SelectionPlaceholder], \[SelectionPlaceholder]]
              \)]},
        {

          ButtonBox[
            \(cyclelowering[\[SelectionPlaceholder],
\[SelectionPlaceholder]]
              \)]}
        },
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      GridDefaultElement:>ButtonBox[ "\\[Placeholder]"]]], NotebookDefault,
  CellMargins->{{Inherited, Inherited}, {5, Inherited}},
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  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
  GroupPageBreakWithin->False,
  CellLabelMargins->{{11, Inherited}, {Inherited, Inherited}},
  DefaultFormatType->DefaultInputFormatType,
  LineSpacing->{1.25, 0},
  AutoItalicWords->{},
  FormatType->InputForm,
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Automatic}, {Automatic, 48}}, WindowFrame->"Palette",
WindowElements->{}, WindowFrameElements->"CloseBox",
WindowClickSelect->False,
ScrollingOptions->{"PagewiseScrolling"->True},
ShowCellBracket->False, CellMargins->{{0, 0}, {Inherited, 0}},
Active->True, CellOpen->True, ShowCellLabel->False,
ShowCellTags->False, ImageMargins->{{0, Inherited}, {Inherited,
0}}, Magnification->1 ]


(***********************************************************************
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***********************************************************************)

(*CellTagsOutline CellTagsIndex->{} *)

(*CellTagsIndex CellTagsIndex->{} *)

(*NotebookFileOutline Notebook[{ Cell[1710, 49, 1663, 54, 138,
NotebookDefault,
  Evaluatable->True,
  CellGroupingRules->"InputGrouping",
  PageBreakAbove->True,
  PageBreakWithin->False,
```

```
  CounterIncrements->"Input"]
} ] *)




(*******************************************************************
End of Mathematica Notebook file.
*******************************************************************)
```

physical consonance's palette

```
(*******************************************************************

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```
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  email: info@wolfram.com
  phone: +1-217-398-0700 (U.S.)

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*******************************************************************)

(*CacheID: 232*)


(*NotebookFileLineBreakTest NotebookFileLineBreakTest*)
(*NotebookOptionsPosition[      3080,         93]*)
(*NotebookOutlinePosition[      4136,        132]*) (*
CellTagsIndexPosition[      4092,        128]*)
(*WindowFrame->Palette*)



Notebook[{ Cell[BoxData[GridBox[{
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        {

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              \[SelectionPlaceholder], \[SelectionPlaceholder]]\)]},
        {

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        {

          ButtonBox[
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              \)]}
        },
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Cached data follows.  If you edit this Notebook file directly, not
using Mathematica, you must remove the line containing CacheID at
the top of the file.  The cache data will then be recreated when
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form's conversion's palette

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                    Mathematica-Compatible Notebook

This notebook can be used on any computer system with Mathematica
3.0, MathReader 3.0, or any compatible application. The data for
the notebook starts with the line of stars above.

To get the notebook into a Mathematica-compatible application, do
one of the following:

* Save the data starting with the line of stars above into a file
  with a name ending in .nb, then open the file inside the application;

* Copy the data starting with the line of stars above to the
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  clipboard, then use the Paste menu command inside the application.

Data for notebooks contains only printable 7-bit ASCII and can be
sent directly in email or through ftp in text mode.  Newlines can
be CR, LF or CRLF (Unix, Macintosh or MS-DOS style).

NOTE: If you modify the data for this notebook not in a
Mathematica- compatible application, you must delete the line
below containing the word CacheID, otherwise
Mathematica-compatible applications may try to use invalid cache
data.

For more information on notebooks and Mathematica-compatible
applications, contact Wolfram Research:
  web: \protect\vrule width0pt\protect\href{http://www.wolfram.com}{http://www.wolf
  email: info@wolfram.com
  phone: +1-217-398-0700 (U.S.)

Notebook reader applications are available free of charge from
Wolfram Research.
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Cached data follows.  If you edit this Notebook file directly, not
using Mathematica, you must remove the line containing CacheID at
the top of the file.  The cache data will then be recreated when
you save this file from within Mathematica.
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